INTRODUCTION TO FACTORING POLYNOMIALS

Introduction

The distributive law is at the heart of factoring; **factoring** is the distributive law used backwards. Recall that the distributive law is

$$a(b+c) = ab + ac\ .$$

However, the equal sign is a two-way street. Looking at the distributive law backwards, we get

$$ab + ac = a(b+c)\ .$$

That's factoring! The factor a is common to both the ab term and the ac term. It can be factored out of the sum of those terms: $ab + ac = a(b+c)$.

In this lesson, we will examine the technique of factoring out the **greatest common factor** of a polynomial. We'll also study the method of **factoring by grouping**. In later lessons, you'll see how these methods turn out to be the key steps in solving equations.

INTRODUCTION TO FACTORING POLYNOMIALS

Terminology

When we write a number as a product of other numbers, we have **factored** the number. The numbers being multiplied are called **factors**. For example, $30 = 2 \cdot 15$, $30 = 6 \cdot 5$, and $30 = 1 \cdot 30$ are three **factorizations** of 30. Each factorization has two factors. Recall that $30 = 2 \cdot 3 \cdot 5$ is the **prime** factorization of 30. This factorization expresses 30 as the product of three prime number factors.

Polynomials can also be factored, and the terminology above also applies to polynomials. To factor a polynomial, we start by factoring out the greatest common factor of all the coefficients in the polynomial.

EXAMPLE A

Factor the polynomial $6x - 15y$.

First, look at the two coefficients, 6 and 15—note that we disregard the sign of the coefficient when we look for the greatest common factor. These numbers both have a common factor of 3. Factor out a 3 from the polynomial:

$$\begin{aligned} 6x - 15y &= 3 \cdot 2x - 3 \cdot 5y \\ &= 3 \cdot (2x - 5y) \\ &= 3(2x - 5y) \end{aligned}$$

To check your work, multiply the polynomial using the distributive law; you should end up with the original polynomial:

$$3(2x + 5y) = 3 \cdot 2x + 3 \cdot 5y = 6x + 15y .$$

Extended Example 1a

Factor the polynomial $20x + 30y - 40z$.

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In addition to factoring out the greatest common factor of all the coefficients, you can also factor out the lowest power of a variable common to each term.

EXAMPLE B

Factor the polynomial $15x^8 - 10x^5 + 25x^2$.

First, look at the coefficients 15, 10, and 25. These numbers all have a common factor of 5. We can factor out a 5 from the polynomial. Notice also that the lowest power of x occurring among the terms is 2. We can also factor out an x^2. We factor out a 5 and an x^2, or in other words, $5x^2$:

$$\begin{aligned} 15x^8 - 10x^5 + 25x^2 &= 5x^2 \cdot 3x^6 - 5x^2 \cdot 2x^3 + 5x^2 \cdot 5 \\ &= 5x^2 \cdot \left(3x^6 - 2x^3 + 5\right) \\ &= 5x^2\left(3x^6 - 2x^3 + 5\right) \end{aligned}$$

Use the distributive law to check (practice doing this step "mentally," meaning without writing it out):

$5x^2$ times $3x^6$ equals $15x^8$

$5x^2$ times $-2x^3$ equals $-10x^5$

$5x^2$ times 5 equals $25x^2$.

Note:

- When multiplying out polynomial expressions, notice that the coefficients are multiplied while the exponents are added.

Extended Example 2a

Factor the polynomial $55x^6 - 20x^4 + 45x^3$.

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EXAMPLE C

Factor: $77x^8y^4z^6 - 33x^{10}y^2z^3$.

First, look at the coefficients 77 and 33. These numbers have a common factor of 11. We can factor out an 11 from the polynomial.

The lowest power of x occurring among the terms is 8.
We can also factor out an x^8.

The lowest power of y occurring among the terms is 2.
We can also factor out a y^2.

The lowest power of z occurring among the terms is 3.
We can also factor out a z^3.

In other words, we'll factor out a $11x^8y^2z^3$:

$$\begin{aligned} 77x^8y^4z^6 - 33x^{10}y^2z^3 &= 11x^8y^2z^3 \cdot 7y^2z^3 - 11x^8y^2z^3 \cdot 3x^2 \\ &= 11x^8y^2z^3 \cdot \left(7y^2z^3 - 3x^2\right) \\ &= 11x^8y^2z^3\left(7y^2z^3 - 3x^2\right) \end{aligned}$$

continued...

Example C, continued…

$77x^8y^4z^6 - 33x^{10}y^2z^3 = 11x^8y^2z^3\left(7y^2z^3 - 3x^2\right)$

Use the distributive law to check the result.

First, check that the coefficients are correct:

$$11x^8y^2z^3\left(7y^2z^3 - 3x^2\right) = 77x^8y^4z^6 - 33x^{10}y^2z^3$$

11 times 7 equals 77

11 times -3 equals -33.

Next, check the x factors:

$$11x^8y^2z^3\left(7y^2z^3 - 3x^2\right) = 77x^8y^4z^6 - 33x^{10}y^2z^3$$

x^8 times (a term with no x factor) equals x^8

x^8 times x^2 equals x^{10}.

Check the y factors:

$$11x^8y^2z^3\left(7y^2z^3 - 3x^2\right) = 77x^8y^4z^6 - 33x^{10}y^2z^3$$

y^2 times y^2 equals y^4

y^2 times (a term with no y factor) equals y^2.

Lastly check the z factors:

$$11x^8y^2z^3\left(7y^2z^3 - 3x^2\right) = 77x^8y^4z^6 - 33x^{10}y^2z^3$$

z^3 times z^3 equals z^6

z^3 times (a term with no z factor) equals z^3.

Again, remember to multiply coefficients and add exponents.

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Extended Example 3a

Factor: $27x^5y^7 - 63x^7y^5$.

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EXAMPLE D

Factor: $5x(3x-7)+2(3x-7)$.

First, notice that this expression consists of the sum of two terms. Recall that terms are held together by "multiplication glue:"

$5 \cdot x \cdot (3x-7)$ is one term and $2 \cdot (3x-7)$ is the other term.

Secondly, notice that each of the two terms has a common factor of $(3x-7)$. We can factor it out :

$$5x \cdot (3x-7)+2 \cdot (3x-7)=(5x+2) \cdot (3x-7)$$

So, we have the factorization:

$$(5x+2)(3x-7)$$

INTRODUCTION TO FACTORING POLYNOMIALS

Question: Factor: $13x(21x-11)+19(21x-11)$.

Question: Factor: $3a(7abc-4)+2(7abc-4)$.

Question: Factor: $12t\left(9t^2u^5+1\right)-17u\left(9t^2u^5+1\right)$.

Factor by Grouping

Whenever you are asked to factor a polynomial with four terms, you may be able to do it in stages. Start by trying to factor the first two terms. Also factor the second two terms, separately. The resulting factorizations may have a common factor that can then be factored out. The previous two sentences describe one case where the **factoring by grouping** method works. This technique is best understood by means of examples.

EXAMPLE E

Factor: $6x^2 - 10x + 21x - 35$.

There is no factor common to all the terms, since 1 is the greatest common factor of all the coefficients and not all terms have an x factor. We could be stuck, or we could try to factor by grouping. To use this method, factor the greatest common factor from the first two terms, and then from the second two terms.

Start by grouping the first two terms and grouping the last two terms:

$$6x^2 - 10x + 21x - 35 = \left(6x^2 - 10x\right) + \left(21x - 35\right)$$

Next, factor the greatest possible factor out of each pair of parentheses:

$$\begin{aligned} 6x^2 - 10x + 21x - 35 &= \left(6x^2 - 10x\right) + \left(21x - 35\right) \\ &= \left(2x \cdot 3x - 2x \cdot 5\right) + \left(7 \cdot 3x - 7 \cdot 5\right) \\ &= 2x \cdot \left(3x - 5\right) + 7 \cdot \left(3x - 5\right) \\ &= 2x\left(3x - 5\right) + 7\left(3x - 5\right) \end{aligned}$$

continued...

INTRODUCTION TO FACTORING POLYNOMIALS

Example E, continued...

$6x^2 - 10x + 21x - 35 = 2x(3x-5) + 7(3x-5)$

Notice that the two terms in $2x(3x-5) + 7(3x-5)$ have a common factor, $(3x-5)$, which we will now factor out:

$$\begin{aligned} 6x^2 - 10x + 21x - 35 &= 2x(3x-5) + 7(3x-5) \\ &= (2x+7)(3x-5) \\ &= (2x+7)(3x-5) \end{aligned}$$

We've factored the polynomial! You should always mentally check that the factorization is correct by using the FOIL method.

Note:

- Always check to see if there's a factor common to all the terms of a polynomial as a first step when factoring!

Extended Example 4a

Factor: $28x^2 - 8x + 35x - 10$.

INTRODUCTION TO FACTORING POLYNOMIALS

There is a very useful (and sometimes necessary) shortcut for factoring a four-term polynomial. Once you factor the first pair of terms, you already know one of the factors of the second pair of terms! In the next set of examples, this short-cut will make the factoring much easier.

EXAMPLE F

Factor: $42x^2 - 12x - 49x + 14$.

After we factor the first two terms, we'll see how the second pair of terms should be factored:

$$42x^2 - 12x - 49x + 14 = (6x \cdot 7x - 6x \cdot 2) - 49x + 14$$
$$= 6x \cdot (7x - 2) - 49x + 14$$
$$= 6x(7x - 2) - 49x + 14$$

One of the factors of the second pair of terms (in the original equation) must be $(7x-2)$. Once you write that down, it will be easy to see the other factor by distributing. $42x^2 - 12x - 49x + 14 = 6x(7x-2) - 49x + 14$

$$= 6x(7x-2) + \boxed{?} \cdot (7x-2)$$

The factor $\boxed{?}$ times $7x$ must equal $-49x$, AND the factor $\boxed{?}$ times -2 must equal 14 . So, the factor $\boxed{?}$ must equal $\boxed{-7}$:

$$42x^2 - 12x - 49x + 14 = 6x(7x-2) - 49x + 14$$
$$= 6x(7x-2) + \boxed{-7} \cdot (7x-2)$$
$$42x^2 - 12x - 49x + 14 = 6x(7x-2) - 7(7x-2)$$

continued...

Example F, continued...

Finally, factor out the common factor $(7x-2)$:

$$\begin{aligned}42x^2-12x-49x+14 &= 6x(7x-2)-7(7x-2)\\ &= (6x-7)(7x-2)\\ &= (6x-7)(7x-2)\end{aligned}$$

Done!

You may want to go through this last example a couple of times until you get a clear understanding of what's happening. Don't worry–it's much easier than it seems at first. You'll have a chance to practice this technique next.

Extended Example 5a

Factor: $9x^2 - 30x - 21x + 70$.

INTRODUCTION TO FACTORING POLYNOMIALS

Remember: ALWAYS check to see if a polynomial's terms all have a common factor. If so, factor out that common factor first! The next few problems combine the techniques of "factoring out the greatest common factor" from the first part of this lesson with "factoring by grouping" from the second part of this lesson.

EXAMPLE G

Factor completely: $40x^2 - 56xz + 60xy - 84yz$.

The greatest common factor of 40, 56, 60, and 84 is 4.
Start by factoring out 4:

$$\begin{aligned} 40x^2 - 56xz + 60xy - 84yz &= 4 \cdot 10x^2 - 4 \cdot 14xz + 4 \cdot 15xy - 4 \cdot 21yz \\ &= 4 \cdot \left(10x^2 - 14xz + 15xy - 21yz\right) \\ &= 4\left(10x^2 - 14xz + 15xy - 21yz\right) \end{aligned}$$

Once you factor out the 4, you can focus entirely on the terms inside the parentheses. Now factor the expression in the parentheses by grouping. Start by factoring the first pair of terms:

$$\begin{aligned} 40x^2 - 56xz + 60xy - 84yz &= 4\left(\left(10x^2 - 14xz\right) + 15xy - 21yz\right) \\ &= 4\left(\left(2x \cdot 5x - 2x \cdot 7z\right) + 15xy - 21yz\right) \\ &= 4\left(2x \cdot \left(5x - 7z\right) + 15xy - 21yz\right) \\ &= 4\left(2x\left(5x - 7z\right) + \boxed{?} \cdot \left(5x - 7z\right)\right) \end{aligned}$$

continued...

Example G, continued...

Since $\boxed{?} \cdot 5x = 15xy$, one sees that $\boxed{?} = \boxed{3y}$.

Also, note that $\boxed{3y} \cdot (-7z) = -21yz$.

Continuing with the factoring by grouping, we get:

$$\begin{aligned} 40x^2 - 56xz + 60xy - 84yz &= 4\left(2x(5x-7z) + 3y(5x-7z)\right) \\ &= 4\left(2x(5x-7z) + 3y(5x-7z)\right) \\ &= 4\left((2x+3y)(5x-7z)\right) \\ &= 4(2x+3y)(5x-7z) \end{aligned}$$

INTRODUCTION TO FACTORING POLYNOMIALS

Extended Example 6a

Factor completely: $10xy - 20xz + 5y^2z - 10yz^2$.

END OF LESSON

Factor as shown in the Lesson: $57cd - 38d^2$

Factor as shown in the Lesson: $c(2d - 3) - 4(3 - 2d)$

Use the methods presented in the lesson to factor each polynomial.
Recall that the order of the factors does not matter,
so that $(\boldsymbol{a} + \boldsymbol{b})(\boldsymbol{c} + \boldsymbol{d}) = (\boldsymbol{c} + \boldsymbol{d})(\boldsymbol{a} + \boldsymbol{b})$.

$5(m - 6n) - x(m - 6n) =$

Use the methods presented in the lesson to factor each polynomial.
Recall that the order of the factors does not matter,
so that $(a+b)(c+d)=(c+d)(a+b)$.

$cd-1+cd^2-d=$

Factor as shown in the Lesson: $m^2+6mn+3m+18n$

Factor as shown in the Lesson: $3x-6+9x^2-18x$

SPECIAL QUADRATIC FACTORIZATIONS

Introduction

In this lesson, we look at factoring special quadratic expressions. They are special cases that should be committed to memory since these cases will come up frequently in the remainder of this course (and the next course). Though the techniques in this lesson may seem a little abstract, they will teach you the key steps in solving equations that will appear later in this chapter.

SPECIAL QUADRATIC FACTORIZATIONS

Quadratic polynomials are polynomials of degree 2 . Quadratic polynomials with one variable can be written in the form $ax^2 + bx + c$, where a, b, and c are constants and $a \neq 0$.

Earlier in this course, you encountered certain special cases when you were learning to multiply polynomials. The following formulas were used as short cuts to multiply binomials (polynomials with two terms):

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$(a+b)(a-b) = a^2 - b^2$$

The equal sign is a two-way street. When we write these equations "backwards," we can use these formulas to factor quadratic polynomials that are in these special forms:

$$a^2 + 2ab + b^2 = (a+b)^2$$

$$a^2 - 2ab + b^2 = (a-b)^2$$

$$a^2 - b^2 = (a+b)(a-b)$$

We will be using these formulas throughout this section. If you learn to use them well, the rest of the course will be easier. For each problem in this section, ask yourself which of these three patterns applies.

SPECIAL QUADRATIC FACTORIZATIONS

EXAMPLE A

Factor completely: $16x^2 + 24x + 9$.

Notice that this quadratic expression has three positive terms. Of the three formulas, only one applies:

$$\rightarrow \boxed{a^2 + 2ab + b^2 = (a+b)^2}$$

$$a^2 - 2ab + b^2 = (a-b)^2$$

$$a^2 - b^2 = (a+b)(a-b)$$

Once you decide which of the three formulas applies, you need to identify the terms in the given polynomial that are represented by a and b. First, rewrite the given quadratic expression in the same form as the chosen formula. The formula starts with a^2, "something" squared.

What must be squared to equal $16x^2$?

Answer: $a^2 = (4x)^2 = 16x^2$, so $a = 4x$.

The formula ends with b^2. (Or "something" squared.)

What must be squared to equal 9?

Answer: $b^2 = 3^2 = 9$, so $b = 3$.

The middle term is then $2 \cdot a \cdot b = 2 \cdot 4x \cdot 3 = 24x$. Our given quadratic expression fits the formula pattern:

$$a^2 \quad + \quad 2 \cdot a \cdot b \quad + \quad b^2 \quad = \quad (a+b)^2$$

$$16x^2 + 24x + 9 \quad = \quad (4x)^2 + 2 \cdot (4x) \cdot 3 + 3^2 \quad = \quad \boxed{(4x+3)^2}$$

SPECIAL QUADRATIC FACTORIZATIONS

Extended Example 1a

Factor completely: $25x^2 - 40x + 16$.

SPECIAL QUADRATIC FACTORIZATIONS

EXAMPLE B

Factor completely: $4x^2 + 36xy + 81y^2$.

Notice that this quadratic expression has three positive terms. We know which formula to use:

$$\rightarrow \boxed{a^2 + 2ab + b^2 = (a+b)^2}$$

$$a^2 - 2ab + b^2 = (a-b)^2$$

$$a^2 - b^2 = (a+b)(a-b)$$

Again, once you decide which of the three formulas applies, you need to find the terms in the given polynomial that are represented by a and b. First, rewrite the given quadratic expression in the same form as the chosen formula. The formula starts with a^2, something squared.

What must be squared to equal $4x^2$?

Answer: $a^2 = (2x)^2 = 4x^2$, so $a = 2x$.

The formula ends with b^2, again something squared.

What must be squared to equal 9?

Answer: $b^2 = (9y)^2 = 81y^2$, so $b = 9y$.

The middle term is then $2 \cdot a \cdot b = 2 \cdot 2x \cdot 9y = 36xy$. Our given quadratic expression fits the formula pattern:

$$a^2 \quad + \; 2 \cdot a \cdot b \; + \quad b^2 \qquad = \quad (a+b)^2$$

$$4x^2 + 36xy + 81y^2 = (2x)^2 + 2 \cdot 2x \cdot 9y + (9y)^2 = (2x+9y)^2$$

SPECIAL QUADRATIC FACTORIZATIONS

Extended Example 2a

Factor completely: $64c^2 - 16cd + d^2$.

SPECIAL QUADRATIC FACTORIZATIONS

EXAMPLE C

Factor completely: $25x^2 - 16y^2$.

Notice that this quadratic expression is the difference of two perfect squares.
There is only one formula to use:

$$a^2 + 2ab + b^2 = (a+b)^2$$

$$a^2 - 2ab + b^2 = (a-b)^2$$

$$\rightarrow \boxed{a^2 - b^2 = (a+b)(a-b)}$$

First, rewrite the given quadratic expression in the same form as the chosen formula. The formula starts with a^2, something squared.

What must be squared to equal $25x^2$?

Answer: $a^2 = (5x)^2 = 25x^2$, so $a = 5x$.

The formula ends with a b^2, again something squared.

What must be squared to equal $16y^2$?

Answer: $b^2 = (4y)^2 = 16y^2$, so $b = 4y$.

Apply the formula with these values for a and b:

$$a^2 \quad - \quad b^2 \quad = \quad (a+b)(a-b)$$

$$25x^2 - 16y^2 = (5x)^2 - (4y)^2 = (5x+4y)(5x-4y)$$

SPECIAL QUADRATIC FACTORIZATIONS

Extended Example 3a

Factor completely: $144y^2 - 49z^2$.

SPECIAL QUADRATIC FACTORIZATIONS

The Sum of Two Squares

We should add one more formula to our list of factoring formulas. Actually, this one is a non-formula!

You have seen that a difference of perfect squares can be factored as $a^2 - b^2 = (a+b)(a-b)$. Unfortunately, the sum of two squares does not factor: $a^2 + b^2 =$ does not factor.* Our list of factoring formulas becomes:

$$a^2 + 2ab + b^2 = (a+b)^2$$

$$a^2 - 2ab + b^2 = (a-b)^2$$

$$a^2 - b^2 = (a+b)(a-b)$$

$$a^2 + b^2 = \text{doesn't factor}$$

The remaining factoring examples for this lesson involve these four formulas.

**Note:*

- Assuming that a and b do not share any common factors.

Note that some problems will require you to factor out the greatest common factor from all the terms as a first step. Again, don't forget to check for such common factors since they should be factored out in your very first step!

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SPECIAL QUADRATIC FACTORIZATIONS

Extended Example 4a

Factor completely: $75R^2 + 12T^2$.

SPECIAL QUADRATIC FACTORIZATIONS

Trinomials that have squares for both end terms look as though they may be the square of some binomial. Alas, most of these polynomials are not the square of any binomial. Always check the middle term.

$$(a+b)^2 = a^2 + 2ab + b^2$$

↑ middle term

EXAMPLE D

Is $16m^2 + 30mn + 9n^2$ the square of a binomial?

Only one formula could apply here:

$$\rightarrow \boxed{a^2 + 2ab + b^2 = (a+b)^2}$$

$$a^2 - 2ab + b^2 = (a-b)^2$$

$$a^2 - b^2 = (a+b)(a-b)$$

$$a^2 + b^2 = \text{doesn't factor}$$

But when we attempt to apply this formula, something goes wrong:

$$a^2 \quad + \quad 2 \cdot a \cdot b \quad + \quad b^2 \quad = \quad (a+b)^2$$

$$16m^2 + 30mn + 9n^2 \overset{?}{=} (4m)^2 + 2\cdot(4m)\cdot(3n) + (3n)^2$$

$$\underbrace{2\cdot(4m)\cdot(3n)}_{24mn}$$

The only possible choice for a and b result in an incorrect middle term! So, the polynomial $16m^2 + 30mn + 9n^2$ is not the square of any binomial.

SPECIAL QUADRATIC FACTORIZATIONS

Extended Example 5a

Is $36x^2 - 30xy + 25y^2$ the square of a binomial?

Sometimes you need to factor out a negative sign in order to see that a factoring formula is applicable. When you factor a negative sign out of a polynomial, each term of the original polynomial switches from positive to negative or from negative to positive:

$$-A+B-C+D=-(A-B+C-D)$$

EXAMPLE E

Factor $-x^2+4xy-4y^2$.

At first it looks like none of the factoring formulas apply. But if you first factor out a negative sign:

$$-x^2+4xy-4y^2=-\left(x^2-4xy+4y^2\right),$$

then we can find an applicable formula to allow us to factor the polynomial in the parentheses:

$$a^2+2ab+b^2=(a+b)^2$$

$$\rightarrow \boxed{a^2-2ab+b^2=(a-b)^2}$$

$$a^2-b^2=(a+b)(a-b)$$

$$a^2+b^2=\text{doesn't factor}$$

Applying this formula, we get:

$$a^2 - 2\cdot a\cdot b + b^2 = (a-b)^2$$

$$-\left(x^2-4xy+4y^2\right)=-\left(x^2-2\cdot x\cdot 2y+(2y)^2\right) = -(x-2y)^2$$

SPECIAL QUADRATIC FACTORIZATIONS

Extended Example 6a

Factor $-x^2 - 14xy - 49y^2$.

END OF LESSON

Factor, if possible: $9s^2 - 16t^2$

Factor, if possible: $ax^2 - 8ax + 16a$

Factor, if possible: $k^2s + 4ks + 4s$

Factor, if possible: $(s+2t)^2+2(s+2t)+1$

Factor the expressions using the formulas on the right, or state that they cannot be factored using the formulas.

Note: You should always factor out the greatest common factor from the entire expression before you apply any of these formulas.

$a^2t^4-14ab^2t^2+49b^4$

$a^2+2ab+b^2=(a+b)^2$
$a^2-2ab+b^2=(a-b)^2$
$a^2-b^2=(a+b)(a-b)$
$a^2+b^2=$ does not factor

Factor completely. (Hint: Factor out a negative sign first!)

$-9y^2-24y-16$

FACTORING QUADRATICS WITH INTEGER COEFFICIENTS

Introduction

This lesson introduces two other methods of factoring – the **Reverse FOIL Method** and the ***AC*-Factoring Method**. We will only consider polynomials and factors with integer coefficients in this section.

FACTORING QUADRATICS WITH INTEGER COEFFICIENTS

Using the FOIL method,

$$\begin{aligned}(x+2)(x+3) &= x^2+3x+2x+2\cdot 3\\ &= x^2+(3+2)x+2\cdot 3\\ &= x^2+\quad 5x \quad +\quad 6.\end{aligned}$$

Notice that the 2 and the 3 were added to get the middle coefficient 5, while they were multiplied to get the end term of 6 on the right.

The **Reverse FOIL Method** is "reverse-engineering" the FOIL method, using the observation above. Let's go through some examples.

EXAMPLE A

Write $x^2+9x+20$ as the product of two binomials.

Suppose that $x^2+9x+20=(x+a)(x+b)$.

Then we know that

$a\cdot b=20$ (because $(x+a)(x+b)=x^2+bx+ax+ab$)

and

$a+b=9$. (because $(x+a)(x+b)=x^2+(a+b)x+ab$)

Can you think of two integers that have a product of 20 and a sum of 9?

Think about the various ways of factoring 20 as the product of two factors while checking to see which two factors add up to 9. The numbers 4 and 5 come to mind since $4\cdot 5=20$ and $4+5=9$. Therefore,

$$x^2+9x+20=(x+4)(x+5).$$

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Extended Example 1a

Write $x^2 + 10x + 21$ as the product of two binomials.

FACTORING QUADRATICS WITH INTEGER COEFFICIENTS

EXAMPLE B

Write $x^2 - 11x + 24$ as the product of two binomials.

This quadratic polynomial could only be the product of two binomials if it could be written in the form:

$$x^2 - 11x + 24 = (x - a)(x - b) .$$

The product of the negative a times the negative b is positive 24 . The two negative terms add up to a negative middle term. As before, we seek a and b such that:

$$a \cdot b = 24$$

and

$$a + b = 11 .$$

Can you think of two integers that have a product of 24 and add up to 11?

Now 6 and 4 have a product of 24 , but they have a sum of 10 . We need factors with sum 11.

And 2 and 12 have a product of 24 , but their sum is 14 . We need the factors to add up to 11.

Aha! 8 and 3 have a product of 24 , <u>and</u> they add up to 11. We need 8 and 3 .

Therefore, we have:

$$x^2 - 11x + 24 = (x - 8)(x - 3)$$

Checking: When factoring, it is always a good idea to check that you factored correctly by multiplying your factors (at least "in your head"). It's easier to multiply than it is to factor. Using the FOIL method, we check our factorization:

$$(x - 8)(x - 3) = x^2 - 3x - 8x + 24$$

$$= x^2 - 11x + 24$$

Here are a couple of additional things to note that might help you with factoring.

Notes

- The order of the factors isn't important, since multiplication is commutative. Either factor may be written first. For example, from the previous screen:

$$x^2 - 11x + 24 = (x-8)(x-3) = (x-3)(x-8)$$

- Sometimes the numbers with the desired product and sum are not immediately obvious. The more factors the numbers have and the larger the factors, the more difficult it is to find the right factorization.
    - There are times when you need to systematically write down <u>all</u> possible factorizations to find the right one, or to show that there is no factorization.

FACTORING QUADRATICS WITH INTEGER COEFFICIENTS

Extended Example 2a

Write $x^2 - 19x + 90$ as the product of two binomials.

EXAMPLE C

Write $x^2 + 5x - 14$ as the product of two binomials.

This quadratic polynomial could be the product of two binomials if it can be written in the form:

$$x^2 + 5x - 14 = (x + a)(x - b).$$

The product of the positive a times the negative b equals negative 14. When we add a positive number to a negative one, we are really subtracting. We seek a and b such that:

$$a \cdot b = 14$$

and

$$a - b = 5.$$

Can you think of two integers that have a product of 14 and a difference of 5?

Aha! 7 and 2 have a product of 14 AND a difference of $7 - 2 = 5$. We need 7 and 2. Therefore,

$$x^2 + 5x - 14 = (x + 7)(x - 2).$$

FACTORING QUADRATICS WITH INTEGER COEFFICIENTS

Note:

- When the last term of a polynomial is <u>positive</u>, as in $x^2 - 12x + 35$, look for two numbers with a product equal to the last term and a **sum** equal to the middle coefficient.

- When the last term of the polynomial is <u>negative</u>, as in $x^2 + 5x - 14$, look for two numbers with a product equal to the last term and a **difference** equal to the middle coefficient.

  - When the last term is negative, also note that:
    - the larger of the two numbers goes with the minus sign if the middle coefficient is negative. For example:

      $$x^2 - 5x - 14 = (x - 7)(x + 2).$$

    - the larger of the two numbers goes with the plus sign if the middle coefficient is positive. For example:

      $$x^2 + 5x - 14 = (x + 7)(x - 2).$$

Extended Example 3a

Write $x^2 - 3x - 18$ as the product of two binomials.

FACTORING QUADRATICS WITH INTEGER COEFFICIENTS

Recap

For the table below, assume that a , b , c , and d are all positive integers.

Polynomial	Factorization	Requirements	Comments
x^2+cx+d	$(x+a)(x+b)$	$a \cdot b = d$ $a+b=c$	Everything is positive.
x^2-cx+d	$(x-a)(x-b)$	$a \cdot b = d$ $a+b=c$	The two negatives add up to a negative middle term. Their product is positive.
x^2+cx-d	$(x+a)(x-b)$	$a \cdot b = d$ $a-b=c$	$a > b$ since the middle term is positive.
x^2-cx-d	$(x+a)(x-b)$	$a \cdot b = d$ $b-a=c$	$a < b$ since the middle term is negative.

FACTORING QUADRATICS WITH INTEGER COEFFICIENTS

When the coefficients of the quadratic polynomial are large, it may be difficult to find the two numbers a and b. Again, sometimes you need to systematically write down all possible factorizations to find the one that works. Despite your efforts, some polynomials simply cannot be factored this way. To prove that a given polynomial cannot be factored, you often need to do an exhaustive and systematic evaluation of all possible values of a and b.

The following simple arithmetic facts can help reduce the guesswork:

- An even number has a factor of 2 hiding within it.
  (For example: $14 = 2 \cdot 7$.)
- The sum (or difference) of two even numbers is even.
  (For example: $4 + 6 = 10$.)
- The sum (or difference) of an even number and an odd number is odd.
  (For example, $4 + 5 = 9$.)

EXAMPLE D

Write $x^2 - 19x + 48$ as the product of two binomials.

For this example we'll use a "brute force" approach. Given its form, we know that this polynomial must factor as $x^2 - 19x + 48 = (x - a)(x - b)$.

The product of the negative a times the negative b is positive 48, and the two negative terms add up to a negative middle term. We seek a and b such that:

$$a \cdot b = 48 \quad \text{and} \quad a + b = 19.$$

continued...

Example D, continued…

We seek two integers that have a product of 48 and a sum of 19. If the numbers do not come to mind. then we can always list all the possible ways that 48 can be written as the product of two numbers. The factors that add up to 19 are the ones we need. Start by finding the prime factorization of 48:

$$48 = 2 \cdot 24 = 2 \cdot 2 \cdot 12 = 2 \cdot 2 \cdot 2 \cdot 6 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$$

$$48 = 2^4 \cdot 3$$

The prime factorization of 48 consists of four twos and one three. Some of these factors multiply out to give a, and the rest multiply to give b. By brute force, we can write down all the possibilities.

$a \cdot b = 48$	$a + b \overset{?}{=} 19$
$1 \cdot 48 = 48$	$1 + 48 = 49$ … too large
$2 \cdot 24 = 48$	$2 + 24 = 26$ … too large
$4 \cdot 12 = 48$	$4 + 12 = 16$ … too small
$8 \cdot 6 = 48$	$8 + 6 = 14$ … too small
$16 \cdot 3 = 48$	$16 + 3 = 19$ … just right!!

The last row of this table shows the factors we need. Therefore,

$$x^2 - 19x + 48 = (x - 16)(x - 3).$$

FACTORING QUADRATICS WITH INTEGER COEFFICIENTS

EXAMPLE E

Write $x^2 - 19x + 48$ as the product of two binomials.

This is the same problem as in Example D. But this time we won't approach it by looking at all possible factorizations of 48 . Instead, we'll try to think like a detective and use deductive reasoning.

We know that $x^2 - 19x + 48 = (x - a)(x - b)$ with

$$a \cdot b = 48 \quad \text{and} \quad a + b = 19.$$

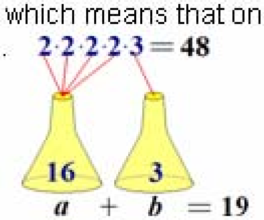
We seek two integers that have a product of 48 and a sum of 19 . As before, start by finding the prime factorization of 48 :

$$48 = 2 \cdot 24 = 2 \cdot 2 \cdot 12 = 2 \cdot 2 \cdot 2 \cdot 6 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$$

$$48 = 2^4 \cdot 3$$

The prime factorization of 48 consists of four twos and one three. Some of these factors multiply out to give us a , and the rest multiply to give us b .

We know that $a + b = 19$. Notice that their sum is an odd number. Recall that an even number plus an even number is an even number. So, both a and b cannot be even. One must be odd, which means that one of the numbers must have all of the (even) four 2 factors.



Let $2^4 = 16 = a$ and then let the remaining factor $3 = b$. Then

$$a \cdot b = 48 \quad \text{and} \quad a + b = 19.$$

We find that $x^2 - 19x + 48 = (x - 16)(x - 3)$.

Extended Example 4a

Write $x^2 - 27x - 160$ as the product of two binomials.

AC -Factoring

The technique called AC **-Factoring** is used when the leading coefficient of the quadratic to be factored isn't 1. Every previous example had a leading coefficient equal to 1 because each polynomial was written with $x^2 = 1 \cdot x^2$. This new technique is called AC -Factoring because of its first step. To factor the polynomial

$$Ax^2 + Bx + C$$

begin by multiplying the first and last coefficients to get the product AC. Then look for two numbers with a product of AC and a sum of B. It is a similar guessing game to the one we have been playing thus far.

As is often the case, this process is best shown by example.

EXAMPLE F

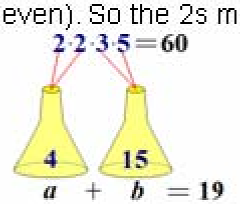
Write $6x^2 + 19x + 10$ as the product of two binomials.

Here $A \cdot C = 60$. We seek two numbers with a product of 60 and a sum of 19. Start by finding the prime factorization of 60:

$$60 = 2 \cdot 30 = 2 \cdot 2 \cdot 15 = 2 \cdot 2 \cdot 3 \cdot 5$$

$$60 = 2^2 \cdot 3 \cdot 5$$

The prime factorization of 60 consists of two 2s, one 3, and a 5. Since the sum of our numbers, 19, is odd, one of our numbers must be even. The other must be odd (else the sum would be even). So the 2s must stay together; one of our numbers has a factor of 4.



continued...

FACTORING QUADRATICS WITH INTEGER COEFFICIENTS

Example F, continued...

What do we do with the 3 and 5? Let the 3 and 5 go to the other factor. One factor is 4 and the other is 15. Then the sum is 19, as desired.

We have found numbers with a product of 60 and sum 19 – namely the numbers 4 and 15. The next step of the AC -Factoring method is to rewrite the middle term of the original polynomial using 4 and 15:

$$6x^2 + 19x + 10 = 6x^2 + 4x + 15x + 10 .$$

Now for a surprising development – factoring-by-grouping works to finish the job:

$$\begin{aligned} 6x^2 + 19x + 10 &= \left(6x^2 + 4x\right) + \left(15x + 10\right) \\ &= 2x(3x+2) + 5(3x+2) \\ &= 2x(3x+2) + 5(3x+2) \\ &= (2x+5)(3x+2) \\ &= (2x+5)(3x+2) \end{aligned}$$

As usual, one can use the FOIL method to double check that the factorization is correct:

$$\begin{aligned} (2x+5)(3x+2) &= 6x^2 + 4x + 15x + 10 \\ &= 6x^2 + 19x + 10 . \end{aligned}$$

FACTORING QUADRATICS WITH INTEGER COEFFICIENTS

Extended Example 5a

Write $10x^2 + 27x - 28$ as the product of two binomials.

FACTORING QUADRATICS WITH INTEGER COEFFICIENTS

As we've mentioned, sometimes quadratic polynomials do not factor. In our next example, we will encounter such a beast and prove that it does not factor as a product of binomials with integer coefficients.

EXAMPLE G

Factor $y^2 + 4y - 16$ as the product of two binomials with integer coefficients.

This quadratic polynomial could be the product of two binomials if it can be written in the form:

$$y^2 + 4y - 16 = (y + a)(y - b).$$

The product of the positive a times the negative b is negative 16. When we add a positive number to a negative one we are really subtracting. We seek a and b such that:

$$a \cdot b = 16$$

and

$$a - b = 4.$$

continued...

Example G, continued...

Can you think of two integers that have a product of 16 and a difference of 4 ? Let's look at all the possible factorizations of 16 as the product of two positive integers:

$a \cdot b = 16$	$a - b \overset{?}{=} 4$
$16 \cdot 1 = 16$	$16 - 1 = 15$ ... too big
$8 \cdot 2 = 16$	$8 - 2 = 6$ ... too big
$4 \cdot 4 = 16$	$4 - 4 = 0$ ... too small

None of these combinations of a and b work. There are no other possible values for a and b. This proves that we are being asked to do the impossible. The quadratic polynomial, $y^2 + 4y - 16$, does not factor. Sad but true. Polynomials like this one that don't factor are sometimes called **prime** (just like integers that don't factor into a product of smaller integers).

Reminder: So far in this lesson, it has not been possible to factor out a greatest common factor from a polynomial first. However, always check for common factors before attempting any other type of factoring.

EXAMPLE H

Factor completely: $3x^4y - 18x^3y - 48x^2y$.

Each coefficient is divisible by 3 and by x^2y . Factoring out $3x^2y$, we get:

$$3x^4y - 18x^3y - 48x^2y = 3x^2y\left(x^2 - 6x - 16\right).$$

We will try to factor the quadratic factor in parentheses, $x^2 - 6x - 16$. This quadratic polynomial could be the product of two binomials if it can be written in the form:

$$x^2 - 6x - 16 = (x + a)(x - b) .$$

The product of the positive a and the negative b is negative 16 . When we add a positive number to a negative one we are really subtracting. We seek a and b such that

$$a \cdot b = 16$$

and

$$b - a = 6 .$$

continued...

Example H, continued...

Can you think of two integers that have a product of 16 and a difference of 6 ? Let's look at all the possible factorizations of 16 as the product of two positive integers:

$a \cdot b = 16$	$b - a \overset{?}{=} 6$
$16 \cdot 1 = 16$	$16 - 1 = 15$ ... too big
$8 \cdot 2 = 16$	$8 - 2 = 6$ ... just right!
$4 \cdot 4 = 16$	$4 - 4 = 0$ ... too small

One can see that the numbers 8 and 2 work. The middle term is negative. The larger number 8 goes with the minus sign in the factorization:

$$x^2 - 6x - 16 = (x+2)(x-8).$$

We now use the above factorization in our original problem:

$$\begin{aligned} 3x^4y - 18x^3y - 48x^2y &= 3x^2y\left(x^2 - 6x - 16\right) \\ &= 3x^2y(x+2)(x-8). \end{aligned}$$

FACTORING QUADRATICS WITH INTEGER COEFFICIENTS

Whenever a trinomial to be factored begins with a negative sign, you need to factor out the negative sign as a first step. Once you factor out the negative, you can ignore it (just make sure you keep copying it out in front, as you go from step to step).

EXAMPLE I

Factor completely: $-x^2 + 14x - 24$.

First we factor out a negative sign (the negatives become positive and the positive becomes negative):

$$-x^2 + 14x - 24 = -\left(x^2 - 14x + 24\right).$$

Now we can factor the trinomial in the parentheses. We need two numbers with product 24 that also add up to 14:

$$= -\left(x^2 - 14x + 24\right) = -(x-a)(x-b).$$

Aha! 12 and 2 do the trick:

$$= -\left(x^2 - 14x + 24\right) = -(x-12)(x-2).$$

FACTORING QUADRATICS WITH INTEGER COEFFICIENTS

Extended Example 6a

Factor completely: $-y^2 - 2y + 99$.

END OF LESSON

Factor, if possible: $x^2 - 5x + 6$

*Factor the expression, where possible. If it is not factorable, write "**Prime**".*

Remember that $(a+b)(c+d) = (c+d)(a+b)$ *(multiplication is commutative) so the order of the factors doesn't matter.*

$4p^2 - 8p - 320 =$
(*Hint:* Factor out a 4 first)

Factor, if possible: $16x^2 - 16x - 12$

*Factor the expression, where possible. If it is not factorable, write "***Prime***".*

Remember that $(a+b)(c+d) = (c+d)(a+b)$ *(multiplication is commutative) so the order of the factors doesn't matter.*

$-x^2 - 13x - 30 =$

SPECIAL CUBIC FACTORIZATIONS

Introduction

In this lesson, we look at factoring **special cubic polynomials**. These are special cases that should be committed to memory as they will come up occasionally in the remainder of this course (as well as the next course). The techniques demonstrated may seem a little abstract but they are key steps in solving equations later in this chapter. In addition, these types of equations actually arise in real-world problems.

SPECIAL CUBIC FACTORIZATIONS

Factoring the Sum and Difference of Two Cubes

Cubic polynomials are polynomials of degree 3 . We will learn to factor the sum and difference of two cubes using these two formulas:

$$a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right)$$
$$a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right).$$

The validity of these factorizations can be demonstrated by multiplying the factors. We multiply the right side of the sum of cubes formula:

$$(a+b)\left(a^2 - ab + b^2\right) = a^3 - a^2b + ab^2 + a^2b - ab^2 + b^3$$
$$= a^3 - \cancel{a^2b} + \cancel{ab^2} + \cancel{a^2b} - \cancel{ab^2} + b^3$$
$$= a^3 + b^3$$

Everything cancels out except the cubed terms. The difference of cubes formula can be validated in the same way.

A Word to the Wise

The quadratic expressions that occur within the parentheses in the cubic factorizations ($a^2 - ab + b^2$ and $a^2 + ab + b^2$), never factor further. Many students have missed a problem by trying to go further than necessary.

SPECIAL CUBIC FACTORIZATIONS

For this lesson, it will help you to know the cubes of the first few positive integers, given below:

$1^3 = 1$	$4^3 = 64$	$7^3 = 343$	$10^3 = 1000$
$2^3 = 8$	$5^3 = 125$	$8^3 = 512$	$11^3 = 1331$
$3^3 = 27$	$6^3 = 216$	$9^3 = 729$	$12^3 = 1728$

EXAMPLE A

Factor completely: $x^3 + 27$.

Since $27 = 3^3$, this is a sum of two cubes: $x^3 + 3^3$. Applying our sum of cubes factoring formula, we get: $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

$$x^3 + 27 = x^3 + 3^3 = (x+3)(x^2 - x \cdot 3 + 3^2)$$

$$x^3 + 27 = (x+3)(x^2 - 3x + 9)$$

EXAMPLE B

Factor completely: $27r^3 + 343z^3$.

Since $27 = 3^3$ and $343 = 7^3$, this is a sum of two perfect cubes: $(3r)^3 + (7z)^3$.

Applying our sum of cubes factoring formula, we get:

$$a^3 + b^3 = (a + b)\left(a^2 - a \cdot b + b^2\right)$$

$$27r^3 + 343z^3 = (3r)^3 + (7z)^3 = (3r + 7z)\left((3r)^2 - 3r \cdot 7z + (7z)^2\right)$$

$$27r^3 + 343z^3 = (3r + 7z)(9r^2 - 21rz + 49z^2).$$

SPECIAL CUBIC FACTORIZATIONS

Extended Example 1a

Factor completely: $y^3 + 125$.

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SPECIAL CUBIC FACTORIZATIONS

EXAMPLE C

Factor completely: $27x^3 - 343y^3$.

Since $27 = 3^3$ and $343 = 7^3$, this is a difference of two perfect cubes:

$$27x^3 - 343y^3 = (3x)^3 - (7y)^3.$$

Applying our difference of cubes factoring formula, we get:

$$a^3 - b^3 = (a - b)\left(a^2 + a \cdot b + b^2\right)$$

$$27x^3 - 343y^3 = (3x)^3 - (7y)^3 = (3x - 7y)\left((3x)^2 + 3x \cdot 7y + (7y)^2\right)$$

$$27x^3 - 343y^3 = (3x - 7y)\left(9x^2 + 21xy + 49y^2\right)$$

Extended Example 2a

Factor completely: $W^3 - 8V^3$.

SPECIAL CUBIC FACTORIZATIONS

EXAMPLE D

Factor completely: $3000p^4 + 81pq^6$.

Both coefficients are divisible by 3. In addition, both terms have a common factor of p. We will factor out their greatest common factor, $3p$:

$$3000p^4 + 81pq^6 = 3p\left(1000p^3 + 27q^6\right).$$

Since $1000 = 10^3$, $27 = 3^3$, and $q^6 = \left(q^2\right)^3$, the expression in the parentheses is the sum of two perfect cubes:

$$1000p^3 + 27q^6 = (10p)^3 + \left(3q^2\right)^3.$$

Applying our sum of cubes factoring formula, we get:

$$a^3 + b^3 = (a + b)\left(a^2 - a \cdot b + b^2\right)$$

$$(10p)^3 + \left(3q^2\right)^3 = \left(10p + 3q^2\right)\left((10p)^2 - 10p \cdot 3q^2 + \left(3q^2\right)^2\right)$$

$$1000p^3 + 27q^6 = \left(10p + 3q^2\right)\left(100p^2 - 30pq^2 + 9q^4\right)$$

Finally, we insert this factorization into our original problem:

$$\begin{aligned} 3000p^4 + 81pq^6 &= 3p\left(1000p^3 + 27q^6\right) \\ &= 3p\left(10p + 3q^2\right)\left(100p^2 - 30pq^2 + 9q^4\right) \\ &= 3p\left(10p + 3q^2\right)\left(100p^2 - 30pq^2 + 9q^4\right). \end{aligned}$$

SPECIAL CUBIC FACTORIZATIONS

Extended Example 3a

Factor completely: $500m^8 - 32m^2n^{12}$.

SPECIAL CUBIC FACTORIZATIONS

Summary

In this lesson, we have added two more formulas to our list of factoring formulas. These will be very useful in the next section, as well as in the following chapter. The complete list is below.

FACTORING FORMULAS

Binomial sum squared:	$a^2 + 2ab + b^2 = (a+b)^2$
Binomial difference squared:	$a^2 - 2ab + b^2 = (a-b)^2$
Sum of squares:	$a^2 + b^2 =$ doesn't factor
Difference of squares:	$a^2 - b^2 = (a+b)(a-b)$
Sum of cubes:	$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
Difference of cubes:	$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

END OF LESSON

Factor the following polynomial.

$p^3 + 512$

Factor the following polynomial.
Hint: First factor out the greatest common factor.

$3x^6 + 81x^3y^3$

One factor of a sum or difference of cubes is given.
Which sum or difference of cubes is it?

$x^2 + 5x + 25$

SOLVING EQUATIONS BY FACTORING

Introduction

In this lesson, we will put our factoring expertise to work to solve equations. A basic idea that we will use is the **zero property of multiplication**:

If $A \cdot B = 0$, then either $A = 0$ or $B = 0$.

SOLVING EQUATIONS BY FACTORING

The Basics of How to Solve by Factoring

To solve equations by factoring, first move all the nonzero terms to one side of the equal sign. We will then have

$$Something = 0\ .$$

Factor that special $Something$ and we will have a product of factors that equals zero:

$$Something = \left(factor_1\right) \cdot \left(factor_2\right) \mathrm{L} \ \left(factor_n\right) = 0\ .$$

Thanks to the zero property of multiplication, we can set each factor equal to zero ($factor = 0$), and then solve each equation individually to find the numbers that make our original equation have a value of zero.

This process converts a difficult problem into a few easy problems!

SOLVING EQUATIONS BY FACTORING

EXAMPLE A

Solve: $(x-2)(x-6)(x+5)=0$.

For this product to equal zero, one of the factors must be zero.

Set each factor equal to zero and solve the resulting linear equations.

In this example, we have three individual equations to solve.

$$x-2=0 \qquad x-6=0 \qquad x+5=0$$

$$\boxed{x=2} \quad \text{or} \quad \boxed{x=6} \quad \text{or} \quad \boxed{x=-5}$$

These three numbers are all solutions to the original equation.

Once an equation is factored into a product of linear factors that equal zero, as you can see, it's fairly easy to solve. The hard work is getting the equation into that form. The most difficult part of this process is factoring the equation.

SOLVING EQUATIONS BY FACTORING

Question: Solve: $(x-4)(x+7)(x+9)=0$.

Question: Solve: $(2x+1)(3x-2)(5x-9)=0$.

Question: Solve: $(9x-7)(3x-8)=0$.

4 of 14

EXAMPLE B

Solve: $x^2 - 13x + 36 = 0$.

This equation is of the form $Something = 0$.

If we can factor the polynomial, we can solve the equation by the zero property of multiplication. A factorization of the polynomial must be of the form

$$x^2 - 13x + 36 = (x - a)(x - b),$$

where $ab = 36$ and $a + b = 13$. The sum of a and b is odd. Therefore, one number is even and the other is odd. Their product must equal 36.

Aha! $4 \cdot 9 = 36$ and $4 + 9 = 13$. We have factored the polynomial:

$$x^2 - 13x + 36 = 0$$

$$(x - 4)(x - 9) = 0$$

A product equals 0 whenever one of its factors is 0:

$$x - 4 = 0 \qquad\qquad x - 9 = 0$$

$$\boxed{x = 4} \quad \text{or} \quad \boxed{x = 9}$$

These two numbers are the solutions to the given equation.

Extended Example 1a

Solve: $x^2 - x - 30 = 0$.

SOLVING EQUATIONS BY FACTORING

Often you will have to do some algebraic juggling to get an equation to the point of factoring. Do not factor expressions that do not equal zero! First, get everything on one side of the equal sign. THEN factor.

EXAMPLE C

Solve: $x(x+1)=42$.

First, we must get into the form $Something = 0$. Subtract 42 from both sides:

$$\begin{array}{rcl} x(x+1) & = & 42 \\ -42 & & -42 \\ \hline x(x+1)-42=0 \end{array}$$

Distribute the x to eliminate the parentheses:

$$x^2+x-42=0.$$

To factor the polynomial, we need two numbers with a product of 42 and a difference of 1. Two consecutive integers with product 42 ... Aha! $7 \cdot 6 = 42$ and $7-6=1$. One of the numbers is positive and the other is negative, since their product is negative. Since the middle coefficient is positive, the larger number, 7, goes with the plus sign. Thus, we have factored the polynomial:

$$x^2+x-42=0$$

$$(x-6)(x+7)=0$$

$$x-6=0 \qquad x+7=0$$

$$\boxed{x=6} \quad \text{or} \quad \boxed{x=-7}$$

Extended Example 2a

Solve: $x(x-4)=32$.

EXAMPLE D

Solve: $2x(2x-17)=2x^2-17x-35$.

First, distribute the $2x$ to eliminate the parentheses:

$$2x(2x-17)=2x^2-17x-35$$
$$4x^2-34x=2x^2-17x-35$$

To get the last equation into the form $Something=0$, move all the terms to the left side of the equal sign:

$$\begin{array}{rcl} 4x^2-34x & = & 2x^2-17x-35 \\ \underline{-2x^2+17x+35} & & \underline{-2x^2+17x+35} \\ 2x^2-17x+35 & = & 0 \end{array}$$

To factor the polynomial, we'll use AC-factoring. Multiply the two end numbers to get 70. We seek two numbers with a product of 70 and a sum of 17. Now, $70=2\cdot5\cdot7$. A moment's thought reveals that $10\cdot7=70$ and $10+7=17$. Both numbers are negative since their product is positive 35, yet they add up to -17. Rewriting the middle term, using 10 and 7, we get:

$$2x^2-17x+35=2x^2-10x-7x+35.$$

continued...

Example D, continued…

$$2x^2 - 17x + 35 = 2x^2 - 10x - 7x + 35$$

Using factoring-by-grouping, this becomes:

$$\begin{aligned} 2x^2 - 17x + 35 &= \left(2x^2 - 10x\right) - 7x + 35 \\ &= 2x(x-5) - 7x + 35 \\ &= 2x(x-5) + \boxed{?} \cdot (x-5) \\ &= 2x(x-5) + \boxed{-7} \cdot (x-5) \\ &= 2x(x-5) - 7(x-5) \\ &= (2x-7)(x-5) \\ &= (2x-7)(x-5) \end{aligned}$$

Thus, we have:

$$2x^2 - 17x + 35 = 0$$
$$(2x-7)(x-5) = 0$$

A product equals 0 whenever one of its factors is 0:

$$2x - 7 = 0 \qquad\qquad x - 5 = 0$$
$$2x = 7 \qquad\qquad x = 5$$
$$\boxed{x = \frac{7}{2}} \quad \text{or} \quad \boxed{x = 5}$$

Extended Example 3a

Solve: $3x(x-3)=2(x+10)$.

SOLVING EQUATIONS BY FACTORING

EXAMPLE E

Solve: $x^2(3x-40)+2=2\left(x^2+1\right)-72x$.

First, distribute to eliminate the parentheses:

$$x^2(3x-40)+2=2\left(x^2+1\right)-72x$$
$$3x^3-40x^2+2=2x^2+2-72x$$

To get it into the form $Something = 0$, move all the terms to the left side of the equal sign:

$$\begin{array}{rcl} 3x^3-40x^2+\quad 2 & = & 2x^2+2-72x \\ -2x^2+72x-2 & & -2x^2-2+72x \\ \hline 3x^3-42x^2+72x & = & 0 \end{array}$$

Each coefficient is divisible by 3 and x . We can factor out $3x$:

$$3x^3-42x^2+72x=0$$
$$3x\left(x^2-14x+24\right)=0$$

continued...

SOLVING EQUATIONS BY FACTORING

Example E, continued...

To factor the polynomial in the parentheses, we will need two numbers with a product of 24 and a sum of 14. A moment's thought reveals that $2 \cdot 12 = 24$ and $2 + 12 = 14$. Both numbers are negative since their product is positive 24, yet they add up to -14:

$$3x\left(x^2 - 14x + 24\right) = 0$$

$$3x(x-2)(x-12) = 0$$

A product equals 0 whenever one of its factors is 0:

$$3x = 0 \qquad x - 2 = 0 \qquad x - 12 = 0$$

$$\boxed{x = 0} \quad \text{or} \quad \boxed{x = 2} \quad \text{or} \quad \boxed{x = 12}$$

These three numbers are the solutions.

Extended Example 4a

Solve: $x^2(5+2x) = 56x - x^2$.

END OF LESSON

Solve the following equation.

$(6y+1)(4y+7)(2y-9)=0; \quad y=?$

Solve the following equation.

$x(5x+23)=10; \quad x=?$

Solve the following equation.

$y^2(3y^2+49)-y^4=21y^3; \quad y=?$