SOLVING SYSTEMS OF LINEAR EQUATIONS BY GRAPHING

Introduction

We have seen how the xy-coordinates of points that satisfy a linear equation form a line in the Cartesian plane. The solution to a system that consists of two linear equations must be the point where the two lines cross, since that point's coordinates satisfy both linear equations.

Systems of linear equations arise in virtually every area of science, engineering, and business. Indeed, a surprising amount of all the computer time used in the world is devoted to solving linear systems. In this lesson, you'll learn how to solve systems of linear equations in two variables by graphing a pair of lines and determining the coordinates of their point of intersection.

SOLVING SYSTEMS OF LINEAR EQUATIONS BY GRAPHING

EXAMPLE A

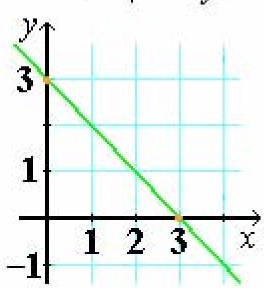
Find x and y such that $x+y=3$ <u>and</u> $x-y=1$.

Plot each equation on the same graph. Find two points for each line. We'll use the x and y-intercepts. Remember: to find the y-intercept, set $x=0$; to find the x-intercept, set $y=0$.

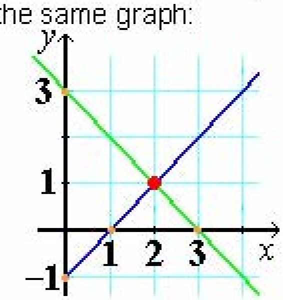
$x+y=3$		x	y
$x+y=3$ $x+0=3$ $x=3$		3	0
$x+y=3$ $0+y=3$ $y=3$		0	3

$x-y=1$		x	y
$x-y=1$ $x-0=1$ $x=1$		1	0
$x-y=1$ $0-y=1$ $y=-1$		0	-1

Plot the first line, $x+y=3$:



Then plot the second, $x-y=1$, on the same graph:



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Example A, continued...

On the graph, it looks like both lines intersect at the point $(2,1)$. Of course, this is only an estimate—it's difficult to see precisely where the point lies. It looks like $(2,1)$, but it could actually be a point very close to $(2,1)$. Solving by graphing is not the most precise method of solving equations. However, we can verify that the point $(2,1)$ is an exact solution to both equations by substituting these x and y values into the original equations.

$$x = 2, \quad y = 1$$

$x + y = 3$	$x - y = 1$
$2 + 1 \stackrel{?}{=} 3$	$2 - 1 \stackrel{?}{=} 1$
$3 = 3$	$1 = 1$

This proves that the point $(2,1)$ satisfies both equations and is the solution to this linear system.

A **system of equations** is two or more equations considered at the same time. A system of two equations is often written in the following form:

$$\begin{cases} y = 2x - 1 \\ y = -x + 5 \end{cases}$$

The solution to such a system is the intersection of the graphs of the equations. The equations don't have to be linear, but in this course they usually will be.

SOLVING SYSTEMS OF LINEAR EQUATIONS BY GRAPHING

Extended Example 1a

Solve this system by graphing: $\begin{cases} y + x = -2 \\ 3y - x = 6 \end{cases}$.

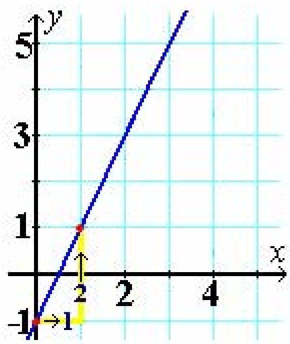
EXAMPLE B

Solve this system by graphing: $\begin{cases} y = 2x - 1 \\ y = -x + 5 \end{cases}$.

Again, we plot both lines on the same graph and estimate their intersection point from the graph. The x, y coordinates of that point give the solution to the linear system.

This time the equations are in slope-intercept form. We will graph them directly, using their slopes and y-intercepts.

The line $y = 2x - 1$ has slope 2 and a y-intercept of $(0,-1)$. Plot the point $(0,-1)$ and use the fact that the $\text{slope} = 2 = \frac{2}{1} = \frac{\text{rise}}{\text{run}}$. If we run 1 and rise 2, we get back to the line:



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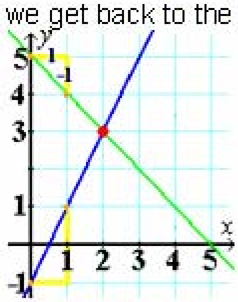
SOLVING SYSTEMS OF LINEAR EQUATIONS BY GRAPHING

Example B, continued...

The line $y = -x + 5$ has slope -1 and a y-intercept of $(0,5)$. Plot the point $(0,5)$, and use the fact that $\text{slope} = -1 = \frac{-1}{1} = \frac{\text{rise}}{\text{run}}$. If we run 1 and fall 1 (because the rise is negative), we get back to the line:



The lines look as if they cross at the point $(2,3)$—where $x = 2$ and $y = 3$.

Let's confirm that $(2,3)$ is the solution:

$y = 2x - 1$	$y = -x + 5$
$3 \overset{?}{=} 2 \cdot 2 - 1$	$3 \overset{?}{=} -2 + 5$
$3 \overset{?}{=} 4 - 1$	$3 = 3$
$3 = 3$	

Indeed, $(2,3)$ is the solution to the linear system.

Note:

- The solution to a system of equations can be specified in two ways. For this example, either as the point $(2,3)$ **or** in the form "$x = 2$ and $y = 3$."

Extended Example 2a

Solve this system by graphing: $\begin{cases} y = x - 2 \\ y = -\dfrac{1}{3}x + 2 \end{cases}$.

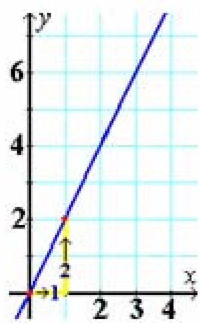
EXAMPLE C

Solve this system by graphing: $\begin{cases} y = 2x \\ y = x + 3 \end{cases}$.

Again, we plot both lines on the same graph and estimate their intersection point from the graph. The x, y coordinates of that point give the solution to the linear system.

This time the equations are in slope-intercept form. We will graph them directly, using their slopes and y-intercepts.

The equation $y = 2x$ can be written as $y = 2x + 0$, so this line has slope 2 and a y-intercept of $(0,0)$, the origin. Plot the origin point $(0,0)$ and use the fact that the slope $= 2 = \frac{2}{1} = \frac{\text{rise}}{\text{run}}$. If we run 1 and rise 2, we get back to the line:



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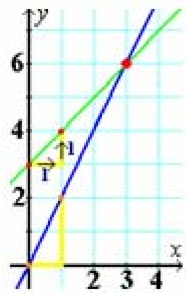
Example C, continued...

The line $y = x + 3$ has slope 1 and y-intercept $(0,3)$. Plot the point $(0,3)$, and use the fact that slope= $1 = \frac{1}{1} = \frac{\text{rise}}{\text{run}}$. If we run 1 and rise 1, we get back to the line:



It looks like the lines cross at the point $(3,6)$—where $x = 3$ and $y = 6$. Let's confirm that $(3,6)$ is the solution:

$y = 2x$	$y = x + 3$
$6 \overset{?}{=} 2 \cdot 3$	$6 \overset{?}{=} 3 + 3$
$6 = 6$	$6 = 6$

Indeed, $(3,6)$ is the solution to the linear system.

SOLVING SYSTEMS OF LINEAR EQUATIONS BY GRAPHING

EXAMPLE D

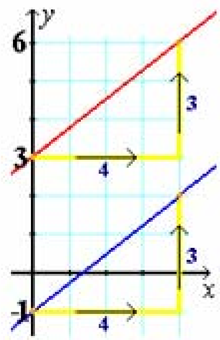
Solve this system by graphing: $\begin{cases} y = \frac{3}{4}x - 1 \\ y = \frac{3}{4}x + 3 \end{cases}$.

Notice that the two lines of this linear system have the same slope, $m = \frac{3}{4}$. The different y-intercepts imply that these lines are parallel. Since parallel lines don't intersect, this system cannot have a solution. There is no need to graph these lines to determine there is no solution. However, if you graph lines that seem to be parallel, check the lines' slopes. Equal slopes imply the lines are either the same line or are distinct parallel lines.

The linear system

$$\begin{cases} y = \frac{3}{4}x - 1 \\ y = \frac{3}{4}x + 3 \end{cases}$$

consists of parallel lines.
It has no solution.



Systems that consist of parallel lines, and therefore have no solution, are called **inconsistent systems**.

10 of 13

SOLVING SYSTEMS OF LINEAR EQUATIONS BY GRAPHING

Extended Example 3a

Solve this system by graphing: $\begin{cases} 3x - y = -3 \\ -3x + y = -3 \end{cases}$.

SOLVING SYSTEMS OF LINEAR EQUATIONS BY GRAPHING

EXAMPLE E

Solve this system by graphing: $\begin{cases} 5x - 2y = 10 \\ 6y - 15x = -30 \end{cases}$.

Plot each equation on the same graph. For each line, find two points on the line. We will use the x and y intercepts. Remember: to find the y-intercept, set $x = 0$. To find the x-intercept, set $y = 0$.

$5x - 2y = 10$	x	y
$5x - 2 \cdot y = 10$ $5x - 2 \cdot 0 = 10$ $5x = 10$ $x = 2$	2	0
$5 \cdot x - 2y = 10$ $5 \cdot 0 - 2y = 10$ $-2y = 10$ $y = -5$	0	-5

$6y - 15x = -30$	x	y
$6 \cdot y - 15x = -30$ $6 \cdot 0 - 15x = -30$ $-15x = -30$ $x = 2$	2	0
$6y - 15 \cdot x = -30$ $6y - 15 \cdot 0 = -30$ $6y = -30$ $y = -5$	0	-5

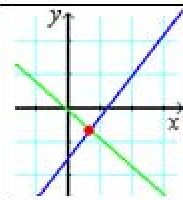
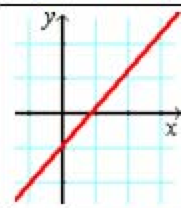
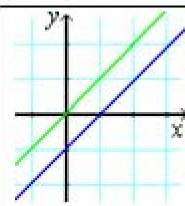
Notice that both of these lines have precisely the same x and y-intercepts. In slope-intercept form, these equations <u>both</u> become $y = \frac{5}{2}x - 5$. We have two equations that result in the same line. Every point on this line satisfies both equations, so even without graphing we know that there are infinitely many solutions to this linear system.

SOLVING SYSTEMS OF LINEAR EQUATIONS BY GRAPHING

Systems that consist of the same line, and therefore have infinitely many solutions, are called **dependent systems**. Since both equations are of the same line, each "depends" on the other.

When the two lines of a linear system intersect at a unique point there is only that one solution. Such systems are said to be **independent systems**.

To summarize, all linear systems consisting of two equations with two variables fall within one of the following three classifications.

Independent systems represent lines that intersect at one point with only one possible solution.	
Dependent systems specify only one line. All the points on this line are solutions. Thus, there are infinitely many solutions.	
Inconsistent systems represent parallel lines which never intersect. There are no solutions.	

END OF LESSON

Solve by graphing: $\begin{cases} y = 3x + 2 \\ 3x + 2y = 4 \end{cases}$

Solve by graphing: $\begin{cases} 3x + 2y = 9 \\ 4.5x + 3y = 13.5 \end{cases}$

Solve the following system by graphing.

$$\begin{cases} 4x + y = 6 \\ 8x - 3y = -3 \end{cases}$$

Solve by graphing: $\begin{cases} -8x + 12y = 6 \\ -4x + 6y = 9 \end{cases}$

Solve the following system by graphing.

$$\begin{cases} 5y = -2x + 1 \\ y = \dfrac{1}{3}x - 2 \end{cases}$$

SOLVING SYSTEMS OF LINEAR EQUATIONS BY SUBSTITUTION

Introduction

This lesson examines the **method of substitution** for solving systems of linear equations. Substitution is not always the most efficient way to solve linear systems, as we'll see in the next section. However, it is more powerful, efficient, and more precise than graphing.

1 of 19

EXAMPLE A

Solve: $\begin{cases} y = 3x - 5 \\ 3y - 4x = 5 \end{cases}$

The first equation states that $y = 3x - 5$. Substitute this expression for y into the second equation and simplify:

$$y = 3x - 5$$

$$\downarrow$$

$$\begin{aligned} 3y - 4x &= 5 \\ 3(3x - 5) - 4x &= 5 \\ 9x - 15 - 4x &= 5 \\ 5x - 15 &= 5 \end{aligned}$$

Now solve for x:

$$\begin{aligned} 5x - 15 &= 5 \\ +15 \quad &\;\; +15 \\ \hline 5x &= 20 \\ \frac{\cancel{5}x}{\cancel{5}} &= \frac{20}{5} \\ \boxed{x = 4} \end{aligned}$$

continued...

SOLVING SYSTEMS OF LINEAR EQUATIONS BY SUBSTITUTION

Example A, continued...

Lastly, substitute this value of $x = 4$ into the original equation, $y = 3x - 5$, and simplify:

$$x = 4$$

$$\downarrow$$

$$y = 3 \cdot x - 5$$

$$y = 3 \cdot 4 - 5$$

$$y = 12 - 5$$

$$\boxed{y = 7}$$

We've found that $x = 4$ and $y = 7$. So, the point $(4, 7)$ is the solution to this system of linear equations. Let's check to see if this point does satisfy both equations of this system:

$y = 3 \cdot x - 5$	$3 \cdot y - 4 \cdot x = 5$
$7 \stackrel{?}{=} 3 \cdot 4 - 5$	$3 \cdot 7 - 4 \cdot 4 \stackrel{?}{=} 5$
$7 \stackrel{?}{=} 12 - 5$	$21 - 16 \stackrel{?}{=} 5$
$7 = 7$	$5 = 5$

The point we found does satisfy both equations!

It's always a good idea to check your solutions in case you made a mistake in your calculations.

SOLVING SYSTEMS OF LINEAR EQUATIONS BY SUBSTITUTION

EXAMPLE B

Solve: $\begin{cases} y = 5 - 2x \\ 3x - 2y = 6 \end{cases}$.

The first equation states that $y = 5 - 2x$. Substitute this expression for y into the second equation, and simplify:

$$y = 5 - 2x$$
$$\downarrow$$
$$3x - 2 \cdot y = 6$$
$$3x - 2 \cdot (5 - 2x) = 6$$
$$3x - 10 + 4x = 6$$
$$7x - 10 = 6$$

Now solve for x:

$$7x - 10 = 6$$
$$+10 \quad +10$$
$$7x = 16$$
$$\frac{7x}{7} = \frac{16}{7}$$
$$\boxed{x = \frac{16}{7}}$$

continued...

SOLVING SYSTEMS OF LINEAR EQUATIONS BY SUBSTITUTION

Example B, continued...

Substitute the value of x you just found into the original equation, $y = 5 - 2x$, and simplify:

$$y = 5 - 2 \cdot x$$

$$y = 5 - 2 \cdot \frac{16}{7}$$

$$y = 5 - \frac{2}{1} \cdot \frac{16}{7}$$

$$y = 5 - \frac{32}{7}$$

Next, we convert 5 to a fraction with 7 as the common denominator. Study this next step because you'll have to do it yourself for future examples:

$$5 = \frac{5}{1} = \frac{5 \cdot 7}{1 \cdot 7} = \frac{35}{7}.$$

Simply replace the 5 with $\frac{35}{7}$:

$$y = 5 - \frac{32}{7}$$

$$y = \frac{35}{7} - \frac{32}{7}$$

$$\boxed{y = \frac{3}{7}}$$

The point $\left(\frac{16}{7}, \frac{3}{7}\right)$ is the solution to this system.

continued...

SOLVING SYSTEMS OF LINEAR EQUATIONS BY SUBSTITUTION

Example B, continued…

Let's check to see if point $\left(\frac{16}{7}, \frac{3}{7}\right)$ satisfies both equations of this system:

$y = 5 - 2 \cdot x$	$3 \cdot x - 2 \cdot y = 6$
$\frac{3}{7} \overset{?}{=} 5 - 2 \cdot \frac{16}{7}$	$3 \cdot \frac{16}{7} - 2 \cdot \frac{3}{7} \overset{?}{=} 6$
$\frac{3}{7} \overset{?}{=} 5 - \frac{32}{7}$	$\frac{48}{7} - \frac{6}{7} \overset{?}{=} 6$
$\frac{3}{7} \overset{?}{=} \frac{35}{7} - \frac{32}{7}$	$\frac{48 - 6}{7} \overset{?}{=} 6$
$\frac{3}{7} \overset{?}{=} \frac{35 - 32}{7}$	$\frac{42}{7} \overset{?}{=} 6$
$\frac{3}{7} = \frac{3}{7}$	$6 = 6$

It checks out—this point is the solution to the system.

Note:

- It is a good idea to check your solutions in this way, especially when fractions are involved!

SOLVING SYSTEMS OF LINEAR EQUATIONS BY SUBSTITUTION

Extended Example 1a

Solve: $\begin{cases} y = 3 + 5x \\ 2x + 4y = 1 \end{cases}$.

7 of 19

SOLVING SYSTEMS OF LINEAR EQUATIONS BY SUBSTITUTION

In the previous examples, one of the original equations of the system expressed one variable in terms of the other (for example, x =something or y =something).

This is the one situation where substitution is the most efficient way to solve a system. Ordinarily, solving for one of the variables is the first step of solving a system of equations by substitution. The variable to solve for is the one that is easier to solve for!

EXAMPLE C

Solve: $\begin{cases} 4x + y = 3 \\ 7x + 3y = 4 \end{cases}$.

Solve the first equation for y since it is the one that is easier to solve for.

$$\begin{aligned} 4x + y &= 3 \\ -4x \quad &\quad -4x \\ \hline y &= -4x + 3 \end{aligned}$$

Substitute this result into the second equation, distribute to remove the parentheses, and combine like terms:

$$y = -4x + 3$$

$$\downarrow$$

$$\begin{aligned} 7x + 3y &= 4 \\ 7x + 3(-4x + 3) &= 4 \\ 7x - 12x + 9 &= 4 \\ -5x + 9 &= 4 \end{aligned}$$

continued...

Example C, continued...

Solve for x:

$$\begin{array}{r} -5x+9 = 4 \\ -9 \quad -9 \\ \hline -5x = -5 \end{array}$$

$$\frac{-5x}{-5} = \frac{-5}{-5}$$

$$\boxed{x = 1}$$

Substitute the x-value into $y = -4x + 3$ to solve for y:

$$x = 1$$

$$\downarrow$$

$$y = -4 \cdot x + 3$$

$$y = -4 \cdot 1 + 3$$

$$y = -4 + 3$$

$$\boxed{y = -1}$$

The solution to the system is $(1, -1)$.

continued...

Example C, continued…

Check $(1,-1)$ in the original equations of the system:

$4 \cdot x + y = 3$	$7 \cdot x + 3 \cdot y = 4$
$4 \cdot 1 + (-1) \overset{?}{=} 3$	$7 \cdot 1 + 3 \cdot (-1) \overset{?}{=} 4$
$4 - 1 \overset{?}{=} 3$	$7 - 3 \overset{?}{=} 4$
$3 = 3$	$4 = 4$

The point $(1,-1)$ does the job! ✓

Note:

- In this last example, we started by finding the variable that was easier to solve for. We could have used either equation and solved for either variable to attain the same solution. However, solving for the easier one makes life simpler!

SOLVING SYSTEMS OF LINEAR EQUATIONS BY SUBSTITUTION

Extended Example 2a

Solve: $\begin{cases} 5x - 12y = 1 \\ 3y - x = 1 \end{cases}$.

SOLVING SYSTEMS OF LINEAR EQUATIONS BY SUBSTITUTION

EXAMPLE D

Solve: $\begin{cases} 3x - 4y = 24 \\ 5x + 6y = 2 \end{cases}$.

Let's solve the first equation for x:

$$\begin{aligned} 3x - 4y &= 24 \\ +4y \quad & \quad +4y \\ \hline 3x &= 4y + 24 \\ \frac{3x}{3} &= \frac{4y + 24}{3} \\ x &= \frac{4y}{3} + \frac{24}{3} \\ x &= \frac{4}{3}y + 8 \end{aligned}$$

Substitute this result for x into the second equation, distribute to remove the parentheses, and combine like terms:

$$\begin{aligned} 5 \cdot x + 6y &= 2 \\ 5\left(\frac{4}{3}y + 8\right) + 6y &= 2 \\ \frac{20}{3}y + 40 + 6y &= 2 \\ \frac{20}{3}y + 40 + \frac{18}{3}y &= 2 \\ \frac{38}{3}y + 40 &= 2 \end{aligned}$$

Note: To get a common denominator of 3, $6y$ became:

$$6y = \frac{6y}{1} = \frac{6y \cdot 3}{3} = \frac{18y}{3}$$

continued...

Example D, continued...

Solve for y:

$$\frac{38}{3}y + 40 = 2$$
$$-40 \quad -40$$
$$\frac{38}{3}y = -38$$
$$\frac{3}{38} \cdot \frac{38}{3}y = \frac{3}{38} \cdot (-38)$$
$$\boxed{y = -3}$$

Substitute the y-value into $x = \frac{4}{3}y + 8$ to solve for x:

$$x = \frac{4}{3} \cdot y + 8$$
$$x = \frac{4}{3} \cdot (-3) + 8$$
$$x = -4 + 8$$
$$\boxed{x = 4}$$

The solution to the system is $(4, -3)$.

continued...

SOLVING SYSTEMS OF LINEAR EQUATIONS BY SUBSTITUTION

Example D, continued…

Check $(4,-3)$ in the original equations of the system:

$3\cdot x-4\cdot y=24$	$5\cdot x+6\cdot y=2$
$3\cdot 4-4\cdot(-3)\overset{?}{=}24$	$5\cdot 4+6\cdot(-3)\overset{?}{=}2$
$12+12\overset{?}{=}24$	$20-18\overset{?}{=}2$
$24=24$	$2=2$

The point $(4,-3)$ fits the bill! ✓

14 of 19

Extended Example 3a

Solve: $\begin{cases} 6x - 7y = 39 \\ 4x - 3y = 21 \end{cases}$.

SOLVING SYSTEMS OF LINEAR EQUATIONS BY SUBSTITUTION

What happens when a linear system is inconsistent (representing parallel lines that never intersect) and does not have a solution?

In this case, the substitution method leads to a contradiction. A **contradiction** is simply a statement that is always false. If you ever arrive at a contradiction such as $1 = 2$ and have made no algebraic errors, then you know that the system does not have a solution. Such a system is inconsistent.

EXAMPLE E

Solve: $\begin{cases} 2x - y = 6 \\ -4x + 2y = 1 \end{cases}$.

Solve the first equation for y since that's the easiest variable to solve for:

$$\begin{array}{rcl} 2x - y &=& 6 \\ -2x & & -2x \\ \hline -y &=& -2x + 6 \\ -1 \cdot (-y) &=& -1 \cdot (-2x + 6) \\ y &=& 2x - 6 \end{array}$$

Substitute this result for y into the second equation and solve:

$$\begin{aligned} -4x + 2y &= 1 \\ -4x + 2 \cdot (2x - 6) &= 1 \\ -4x + 4x - 12 &= 1 \\ \boxed{-12 = 1} & \quad \text{Contradiction!} \end{aligned}$$

We arrived at a contradiction, which signifies that this linear system has no solution. The system is inconsistent; the lines of the system are parallel.

16 of 19

SOLVING SYSTEMS OF LINEAR EQUATIONS BY SUBSTITUTION

What happens when a linear system is dependent (the equations in the system represent the same line), having infinitely many solutions?

In this case, the substitution method leads to a tautology. A **tautology** is a statement that is ALWAYS true. If you ever arrive at a tautology, such as $5 = 5$, and have made no algebraic errors, the system is dependent. It consists of equations of the same line. Every point on this line is a solution to the system.

EXAMPLE F

Solve: $\begin{cases} 2x - y = 6 \\ -4x + 2y = -12 \end{cases}$

Solve the first equation for y since that's the easiest variable to solve for:

$$\begin{aligned} 2x - y &= 6 \\ -2x \qquad &\;\; -2x \\ \hline -y &= -2x + 6 \\ -1\cdot(-y) &= -1\cdot(-2x+6) \\ y &= 2x - 6 \end{aligned}$$

continued...

Example F continued ...

Substitute $y = 2x - 6$ into the second equation and solve (distribute to remove the parentheses, and combine like terms):

$$y = 2x - 6$$
$$\downarrow$$
$$-4x + 2y = -12$$
$$-4x + 2 \cdot (2x - 6) = -12$$
$$-4x + 4x - 12 = -12$$
$$\boxed{-12 = -12}$$
$$\text{Tautology!}$$

We arrived at a tautology, which signifies that this linear system is dependent. The system consists of different equations of the same line, of which every point is a solution. There are infinitely many solutions, one for each point on the line.

SOLVING SYSTEMS OF LINEAR EQUATIONS BY SUBSTITUTION

Extended Example 4a

Solve: $\begin{cases} y = \dfrac{2}{3}x - 5 \\ 9y - 6x = 7 \end{cases}$.

END OF LESSON

Solve by substitution: $\begin{cases} v = 1 - u \\ u + 2v = 5 \end{cases}$

Use substitution to solve the system of linear equations.

$$\begin{cases} y = 1 - x \\ x + 2y = 5 \end{cases}$$

Solve by substitution: $\begin{cases} 2x + y = 5 \\ 4x - 3y = -7 \end{cases}$

At a rock concert, the cost for tickets was $5 for students and $8 for others. There were 542 paid admissions, and the gate receipts totaled $3,286. How many students were at the concert? Create a system of equations and solve by substitution.

Use substitution to solve the system of linear equations.

$$\begin{cases} 3x + y = -8 \\ 2x - 4y = 7 \end{cases}$$

Use substitution to solve the system of linear equations.

$$\begin{cases} y = 10x \\ 7x - 9y = -1 \end{cases}$$

SOLVING SYSTEMS OF LINEAR EQUATIONS BY ELIMINATION

Introduction

The method of substitution can get a bit tedious at times, as you've seen. Luckily, there's an easier way to solve difficult cases! In this section, we will study the **method of elimination** (sometimes called the **addition method**).

1 of 18

EXAMPLE A

Solve: $\begin{cases} 2x+3y=-5 \\ 4x-3y=17 \end{cases}$.

In each equation, the left side equals the right side. So, the sum of the left sides of the two equations should equal the sum of the right sides of the two equations:

$$\begin{array}{r} 2x+3y=-5 \\ \underline{4x-3y=17} \\ 6x \quad\quad =12 \end{array}$$

Something interesting happens when we add these equations—adding these equations eliminates the y variable. We can easily solve the resulting equation for x:

$$6x=12$$

$$\frac{6x}{6}=\frac{12}{6}$$

$$\frac{\cancel{6}x}{\cancel{6}}=2$$

$$\boxed{x=2}$$

continued...

Example A, continued…

Substituting $x = 2$ into the first equation yields:

$$x = 2$$

$$\downarrow$$

$$2 \cdot x + 3y = -5$$

$$2 \cdot 2 + 3y = -5$$

$$4 + 3y = -5$$

$$\underline{-4 \qquad\quad -4}$$

$$3y = -9$$

$$\frac{3y}{3} = \frac{-9}{3}$$

$$\boxed{y = -3}$$

The solution to the system is $x = 2$ and $y = -3$ or, in other words, the point $(2, -3)$ is the solution.

Remember, it's always a good idea to check your solutions in case you made a mistake in your calculations, even though we won't show the checking step in this lesson.

Extended Example 1a

Solve: $\begin{cases} 2x - 7y = -4 \\ 3x + 7y = -6 \end{cases}$.

The method used in the examples above worked because one of the variables was eliminated easily by adding the equations. This will only work when the variables have coefficients that are opposites of each other. Few systems happen to be in such a convenient form. Luckily, it's not difficult to get equations into this form.

EXAMPLE B

Solve: $\begin{cases} 5x + 4y = 1 \\ -3x - 2y = 1 \end{cases}$.

Study the two equations in the system. Is there a number we could multiply the second equation by to get a y term that is opposite the y term, $4y$, in the first equation? We could multiply the second equation by 2 to get $-4y$. To multiply the second equation by 2, we double each coefficient in the second equation. Then, we add the resulting equations:

$$\begin{array}{lcl} 5x + 4y = 1 & & 5x + 4y = 1 \\ -3x - 2y = 1 & \xrightarrow{\cdot 2} & \underline{-6x - 4y = 2} \\ & & -x = 3 \quad \text{or} \quad x = -3 \end{array}$$

Substitute $x = -3$ into the first equation and solve for y:

$$\begin{aligned} 5 \cdot x + 4y &= 1 \\ 5 \cdot (-3) + 4y &= 1 \\ -15 + 4y &= 1 \\ +15 \qquad &\; +15 \\ 4y &= 16 \\ y &= 4 \end{aligned}$$

The solution is $(-3, 4)$.

EXAMPLE C

Solve: $\begin{cases} 2x+5y=-3 \\ 4x+9y=11 \end{cases}$.

If we multiply the first equation by -2, we'll be able to eliminate the x terms by adding the resulting equations:

$$\begin{array}{lcr} 2x+5y=-3 & \xrightarrow{\cdot(-2)} & -4x-10y=6 \\ 4x+9y=11 & & \underline{4x+9y=11} \\ & & -y=17 \\ & & \boxed{y=-17} \end{array}$$

Substitute $y=-17$ into the (original) first equation and solve for x:

$$y=-17$$
$$\downarrow$$
$$\begin{aligned} 2x+5\cdot y &= -3 \\ 2x+5\cdot(-17) &= -3 \\ 2x-85 &= -3 \\ \underline{+85} &\quad \underline{+85} \\ 2x &= 82 \\ \boxed{x=41} \end{aligned}$$

The solution is $(41,-17)$.

SOLVING SYSTEMS OF LINEAR EQUATIONS BY ELIMINATION

Extended Example 2a

Solve: $\begin{cases} -3x+7y=-53 \\ 15x+11y=35 \end{cases}$.

SOLVING SYSTEMS OF LINEAR EQUATIONS BY ELIMINATION

Sometimes it is necessary to multiply each equation by a different number to eliminate one of the variables. This process is similar to "finding a common denominator."

EXAMPLE D

Solve: $\begin{cases} 3x + 7y = 67 \\ 2x + 9y = 101 \end{cases}$.

We can multiply the first equation by -2 and multiply the second equation by 3, to get opposite x terms. Then we add the resulting equations:

$$3x + 7y = 67 \xrightarrow{\cdot(-2)} -6x - 14y = -134$$
$$2x + 9y = 101 \xrightarrow{\cdot 3} \underline{6x + 27y = 303}$$
$$13y = 169$$

Solve for y:

$$13y = 169$$
$$\frac{13y}{13} = \frac{169}{13}$$
$$\boxed{y = 13}$$

continued...

Example D, continued…

Substitute $y = 13$ into the (original) first equation and solve for x:

$$y = 13$$
$$\downarrow$$
$$\begin{aligned} 3x + 7 \cdot y &= 67 \\ 3x + 7 \cdot 13 &= 67 \\ 3x + 91 &= 67 \\ -91 \quad & \;\; -91 \\ \hline 3x &= -24 \\ \boxed{x = -8} \end{aligned}$$

The solution is $(-8, 13)$.

In this last example, we eliminated the variable x, but we could have chosen to eliminate y. In the next example, it is easier to eliminate y because it involves working with smaller numbers.

EXAMPLE E

Solve: $\begin{cases} 5x + 2y = 38 \\ 11x - 3y = 91 \end{cases}$.

We'll eliminate y by multiplying the first equation by 3 while multiplying the second equation by 2. Add the resulting equations:

$$\begin{array}{rcr} 5x + 2y = 38 & \xrightarrow{\cdot 3} & 15x + 6y = 114 \\ 11x - 3y = 91 & \xrightarrow{\cdot 2} & \underline{22x - 6y = 182} \\ & & 37x = 296 \end{array}$$

Solving for x, we get:

$$37x = 296$$
$$\frac{37x}{37} = \frac{296}{37}$$
$$\boxed{x = 8}$$

continued...

Example E, continued...

Substitute $x = 8$ into the first equation and solve for y :

$$x = 8$$
$$\downarrow$$
$$5 \cdot x + 2y = 38$$
$$5 \cdot 8 + 2y = 38$$
$$40 + 2y = 38$$
$$-40 \qquad -40$$
$$2y = -2$$
$$\frac{2y}{2} = \frac{-2}{2}$$
$$\boxed{y = -1}$$

The solution is $(8, -1)$.

SOLVING SYSTEMS OF LINEAR EQUATIONS BY ELIMINATION

Extended Example 3a

Solve: $\begin{cases} 11x + 6y = 95 \\ 13x - 15y = 289 \end{cases}$.

12 of 18

SOLVING SYSTEMS OF LINEAR EQUATIONS BY ELIMINATION

As with the method of substitution, contradictions (false statements) imply that a system is inconsistent (representing parallel lines that never intersect), and tautologies (true statements) imply that a system is dependent (the equations in the system represent the same line).

EXAMPLE F

Solve: $\begin{cases} 5x - 3y = 12 \\ \frac{5}{3}x - y = 4 \end{cases}$.

Multiply the first equation by –1 and the second equation by 3. Then add the resulting equations:

$$\begin{array}{lcr} 5x - 3y = 12 & \xrightarrow{\cdot(-1)} & -5x + 3y = -12 \\ \frac{5}{3}x - y = 4 & \xrightarrow{\cdot 3} & \underline{5x - 3y = 12} \\ & & 0 = 0 \end{array}$$

Tautology!

This tautology implies that the system is dependent. Both equations are of the same line. Every point on the line is a solution, and, hence, there are infinitely many solutions.

SOLVING SYSTEMS OF LINEAR EQUATIONS BY ELIMINATION

EXAMPLE G

Solve: $\begin{cases} 5x - 3y = 12 \\ \dfrac{5}{3}x - y = 3 \end{cases}$.

Multiply the first equation by –1 and the second equation by 3. Then add the resulting equations:

$$\begin{aligned} 5x - 3y = 12 \quad &\xrightarrow{\cdot(-1)} \quad -5x + 3y = -12 \\ \frac{5}{3}x - y = 3 \quad &\xrightarrow{\cdot 3} \quad \underline{5x - 3y = \quad 9} \\ & \qquad\qquad\qquad 0 = -3 \end{aligned}$$

Contradiction!

The contradiction implies that this system is inconsistent. The equations of the linear system are of parallel lines. The lines don't intersect, and so there is no solution.

Question: Solve: $\begin{cases} 2x - 7y = 14 \\ \dfrac{2}{7}x - y = 2 \end{cases}$.

SOLVING SYSTEMS OF LINEAR EQUATIONS BY ELIMINATION

Question: Solve: $\begin{cases} 2x - 7y = 14 \\ \dfrac{2}{7}x - y = 3 \end{cases}$.

16 of 18

SOLVING SYSTEMS OF LINEAR EQUATIONS BY ELIMINATION

Question: Solve: $\begin{cases} -6x + 5y = -10 \\ \dfrac{12}{5}x - 2y = 4 \end{cases}$

Summary: Solving Systems of Linear Equations

In this chapter, we have examined three methods for solving systems of linear equations.

1. **Graphing**: Graph the two lines and find the point of intersection. This is usually quite an inconvenient (and tedious) approach, and can only give crude approximations to the actual solution.

2. **Substitution**: Solve one of the equations for a variable. Substitute that value into the other equation and solve for the other variable. This method always yields exact solutions, but the algebra often gets messy (fractions!). It is best to use substitution when one of the equations of the system has one variable in terms of the other variable.

3. **Elimination**: Multiply each equation by values such that, when the two equations are added together, one of the variables will be eliminated. Substitute the known value into either equation to solve for the remaining variable. This is generally the simplest technique to use to solve systems of linear equations.

Solve by elimination: $\begin{cases} 4J + 3K = 0 \\ 9J - 2K = -35 \end{cases}$

Use elimination to solve the system of linear equations.

$$\begin{cases} 4x + 3y = 0 \\ 9x - 2y = -35 \end{cases}$$

Solve by elimination: $\begin{cases} 8x - 7y = 12 \\ 5x + 3y = 4 \end{cases}$

Use elimination to solve the system of linear equations.

$$\begin{cases} 8x + 3y = 5 \\ 5x - 7y = -6 \end{cases}$$

Use elimination to solve the system of linear equations.

$$\begin{cases} \frac{2}{3}x + \frac{5}{3}y = 2 \\ \frac{1}{3}x - \frac{4}{3}y = 1 \end{cases}$$

APPLICATIONS OF LINEAR SYSTEMS

Introduction

Systems of linear equations can be used to solve many problems in almost all areas of human endeavor. In this section, you'll get a lot of practice translating from English to algebra as you solve linear systems to address real-world problems. When defining a variable for your problem, use a letter that helps you remember what that variable represents.

EXAMPLE A

The sum of Ali's and Mark's ages is 48. Ali is 12 years older than Mark. How old are they?

Let $A=$ Ali's age and $M=$ Mark's age.

The information given above translates into this system of equations:

$$\begin{cases} A+M=48 \\ A=12+M \end{cases}$$

To solve this linear system, substitute the second equation into the first, and solve for M:

$$A=12+M$$

$$\downarrow$$

$$\begin{aligned} A + M &= 48 \\ 12+M + M &= 48 \\ 12+2M &= 48 \\ -12 \qquad & \quad -12 \\ \hline 2M &= 36 \\ \frac{2M}{2} &= \frac{36}{2} \\ \boxed{M = 18} \end{aligned}$$

Mark is 18 years old. Recall that we are told that Ali is 12 years older than Mark. So, Ali is $18+12=30$ years old.

For the following Extended Examples, use the method of substitution to solve the linear systems you define.

Extended Example 1a

The sum of Marie's and Katie's ages is 92. Marie is 30 years older than Katie. How old are they?

APPLICATIONS OF LINEAR SYSTEMS

EXAMPLE B

A jar contains 40 coins consisting entirely of nickels and dimes, worth a total of \$2.80. How many nickels and how many dimes are in the jar?

Let the N = number of nickels and D = the number of dimes. The total number of dimes and nickels is $N+D=40$.

We have N nickels, worth 5 cents each. So, we have a total of $5N$ cents worth of nickels. We have D dimes, worth 10 cents each. So, we have a total of $10D$ cents worth of dimes. The coins are worth a total of 280 cents, and so $5N+10D=280$.

Therefore, we must solve this linear system: $\begin{cases} N+D=40 \\ 5N+10D=280 \end{cases}$.

Notice that all the numbers in the second equation are divisible by 5. Divide the second equation by 5. Also, multiply the first equation by −1 which, with the division by 5, will allow us to eliminate N :

$$N+D=40 \xrightarrow{\cdot(-1)} -N-D=-40$$

$$5N+10D=280 \xrightarrow{\div 5} N+2D=56$$

$$\boxed{D=16}$$

There are 16 dimes. Substitute this into the (original) first equation to solve for N :

$$\begin{aligned} N+D &= 40 \\ N+16 &= 40 \\ -16 \quad & \quad -16 \\ \hline \boxed{N =} & \boxed{24} \end{aligned}$$

There are 24 nickels and 16 dimes in the jar.

APPLICATIONS OF LINEAR SYSTEMS

For the following Extended Examples, use the method of elimination to solve the linear systems you define.

Extended Example 2a

A jar contains 54 coins consisting entirely of nickels and quarters, worth a total of $8.90. How many nickels and how many quarters are in the jar?

EXAMPLE C

A total of 444 movie tickets are sold, bringing in a total of \$3,948. Two types of tickets were sold: adult tickets were sold for \$10 each and child tickets were sold for \$6 each. How many tickets of each type were sold?

Let A = the number of adult tickets and C = the number of child tickets.

The total number of tickets sold is $A+C=444$.

A adult tickets were sold, worth \$10 each, for a total of $10A$ dollars worth of adult tickets.

C child tickets were sold, worth \$6 each, for a total of $6C$ dollars worth of child tickets.

Ticket sales totaled \$3,948. So: $10A+6C=3948$.

The linear system to solve is: $\begin{cases} A+C=444 \\ 10A+6C=3948 \end{cases}$.

Notice that all the numbers in the second equation are divisible by 2. Divide the second equation by 2 (this is to simply make the numbers smaller and easier to work with– the smaller the better!):

$$\begin{array}{lcl} A+C=444 & & A+C=444 \\ 10A+6C=3948 & \xrightarrow{\div 2} & 5A+3C=1974 \end{array}$$

continued...

Example C, continued ...

Next, multiply the first equation by -3 . We can then eliminate C by adding the resulting equations:

$$\begin{array}{lcr} A + C = 444 & \xrightarrow{\cdot(-3)} & -3A - 3C = -1332 \\ 5A + 3C = 1974 & & \underline{5A + 3C = 1974} \\ & & 2A = 642 \end{array}$$

Solve for A:

$$\frac{2A}{2} = \frac{642}{2}$$

$$\boxed{A = 321}$$

Substitute this into the first equation and solve for C:

$$A = 321$$

$$\downarrow$$

$$\begin{aligned} A + C &= 444 \\ 321 + C &= 444 \\ -321 \quad & \quad -321 \end{aligned}$$

$$\boxed{C = 123}$$

321 adult tickets and 123 child tickets were sold.

APPLICATIONS OF LINEAR SYSTEMS

For the following Extended Examples, use the method of elimination to solve the linear systems you define.

Extended Example 3a

A total of 258 tickets are sold to a concert, bringing in a total of $2,640. Two types of tickets were sold: adult tickets were sold for $12 each and student tickets were sold for $8 each. How many tickets of each type were sold?

APPLICATIONS OF LINEAR SYSTEMS

EXAMPLE D

A plane takes off at 9:00 AM, traveling north at 230 miles per hour. A jet takes off at 11:00 AM, traveling north at 520 miles per hour. At what time will the jet overtake the plane? (Round your answer to the nearest minute.)

When the jet takes off, the plane has been flying for 2 hours, and has gone a distance, d :

$$d = \left(230\frac{\text{miles}}{\text{hour}}\right)\cdot\left(2\text{ hours}\right) = 460\text{ miles}.$$

The plane continues to fly at 230 miles per hour. Thus, after t hours (after 11:00 AM) it has flown an additional:

$$\left(230\frac{\text{miles}}{\text{hour}}\right)\cdot\left(t\text{ hours}\right) = 230t\text{ miles}.$$

If t is the number of hours past 11:00 AM, the plane will have traveled a total of

$$d = 460 + 230t,$$

measured in miles. Meanwhile, the jet will have traveled at 520 miles an hour for t hours and will have traveled a total distance of

$$d = 520t.$$

We use the same d for both equations because at the moment the jet overtakes the plane, they will have flown exactly the same distance, d .
We have this linear system:

$$\begin{cases} d = 460 + 230t \\ d = 520t \end{cases}.$$

continued...

Example D, continued…

To solve this system, substitute the second equation into the first:

$$d = 520t$$

$$\downarrow$$

$$d = 460 + 230t$$

$$520t = 460 + 230t$$

$$-230t \qquad -230t$$

$$290t = 460$$

$$\frac{290t}{290} = \frac{460}{290}$$

$$t = \frac{46}{29}$$

Remember, this is the number of hours. Converting this to a decimal, we get:

$$t = \frac{46}{29} \cong 1.586206897 \text{ hours.}$$

That is, we get 1 hour and an additional 0.586206897 hour. Converting the fractional hour to minutes (just multiply by 60 since there are 60 minutes in each hour), we get:

$$(0.586206897 \text{ hours}) \cdot \left(60 \; \frac{\text{minutes}}{\text{hours}}\right) = 35.17241382 \text{ minutes}.$$

Rounding to the nearest minute, the jet will take 1 hour and 35 minutes to overtake the plane. Since the jet started at 11:00 AM, the jet will overtake the plane at about 12:35 PM.

APPLICATIONS OF LINEAR SYSTEMS

EXAMPLE E

A solar heating system costs $23,000 to install with a yearly operating cost of $200. An electric heating system costs $5,000 to install but has a yearly operating cost of $1,600. How many years will pass before the total cost of the solar heating system is less than the total cost of the electric heating system?

Let t be the number of years and let C be the total cost (after t years).

Solar: After the initial cost of $23,000, each year adds another $200 to the total cost. After t years, the cost of the solar heating system is given by:

$$C = 23000 + 200t\ .$$

Electric: After the initial cost of $5,000, each year adds another $1,600 to the total cost. After t years, the cost of the electric heating system is given by:

$$C = 5000 + 1600t\ .$$

Let's find out how many years will pass before the two costs are equal. We must solve the system:

$$\begin{cases} C = 23000 + 200t \\ C = 5000 + 1600t \end{cases}.$$

continued...

Example E, continued...

Substitute the first equation into the second, and solve for t:

$$C = 23000 + 200t$$

$$\downarrow$$

$$\begin{aligned}
C &= 5000 + 1600t \\
23000 + 200t &= 5000 + 1600t \\
-200t \quad & \quad\; -200t \\
\hline
23000 &= 5000 + 1400t \\
-5000 \quad & -5000 \\
\hline
18000 &= 1400t \\
\frac{18000}{1400} &= \frac{1400t}{1400} \\
\frac{18000}{1400} &= t \\
\boxed{\frac{90}{7} = t}
\end{aligned}$$

Converting this to an approximate decimal, we find $t = \frac{90}{7} \cong 12.857$ — well over 12 years. It will be about 13 years before the total cost of the solar heating system is less than the total cost of the electric heating system.

EXAMPLE F

A salesperson applying for work is considering two job offers. Company A offers to pay $100 per day plus 6% of the total sales for that day. Company B offers $70 per day plus 10% of total daily sales. What amount of daily sales will make Company B as lucrative for the salesperson as Company A?

Let s be the daily sales total, and let E be the salesperson's daily earnings.

Company A: 6% of s is added to the fixed salary of $100. Since "6% of s" = $0.06s$, the salesperson's earnings would be:

$$E = 100 + 0.06s.$$

Company B: 10% of s is added to the fixed salary of $70. Since "10% of s" = $0.10s$, the salesperson's earnings would be:

$$E = 70 + 0.10s.$$

For what amount of sales will these earnings be equal? We must solve the system:

$$\begin{cases} E = 100 + 0.06s \\ E = 70 + 0.10s \end{cases}.$$

continued...

Example F, continued...

Substitute the first equation into the second, and solve for s:

$$E = 100 + 0.06s$$

$$\downarrow$$

$$\begin{aligned} E &= 70 + 0.10s \\ 100 + 0.06s &= 70 + 0.10s \\ -0.06s \quad & \quad -0.06s \\ \hline 100 &= 70 + 0.04s \\ -70 \quad & -70 \\ \hline 30 &= 0.04s \\ \frac{30}{0.04} &= \frac{0.04s}{0.04} \\ \boxed{750 = s} \end{aligned}$$

The daily sales would have to be $750 for Company B to pay as much as Company A.

For the following Extended Examples, use the method of substitution to solve the linear systems you define.

Extended Example 4a

A salesperson applying for work is considering two job offers. Company A offers to pay $80 per day plus 9% of the total sales for that day. Company B offers $90 per day plus 8% of total daily sales. What amount of daily sales will make Company A as lucrative for the salesperson as Company B?

EXAMPLE G

You have a 70% acid solution and a 10% acid solution. How many liters of each must you mix to obtain 5 liters of 30% acid solution? (Round your answers to the nearest milliliter–the nearest thousandth of a liter.)

Let S be the number of liters of 70% acid solution that you must mix with T liters of 10% acid solution, to end up with 5 liters of 30% acid solution. The number of liters of the mixed solution equals the total number of liters of solution needed:

$$S+T=5.$$

Also, the amount of acid you start with equals the amount of acid you end up with. S liters of 70% acid contain $0.70 \cdot S$ liters of acid. T liters of 10% acid contain $0.10 \cdot T$ liters of acid. 5 liters of 30% acid solution contain $0.30 \cdot 5 = 1.5$ liters of acid. We must have:

$$0.70S + 0.10T = 1.5.$$

Put these two equations together and we have a linear system to solve:

$$\begin{cases} S+T=5 \\ 0.70S+0.10T=1.5 \end{cases}.$$

It's easiest to eliminate the decimals first by multiplying the second equation by 10:

$$\begin{array}{rcl} S+T=5 & \longrightarrow & S+T=5 \\ 0.70S+0.10T=1.5 & \xrightarrow{\cdot 10} & 7S+T=15 \end{array}$$

continued...

Example G, continued...

Eliminate T by multiplying the top equation by -1 then adding the two equations:

$$\begin{array}{rcr} S+T=5 & \xrightarrow{\cdot(-1)} & -S-T=-5 \\ 7S+T=15 & \longrightarrow & 7S+T=15 \\ \hline & & 6S=10 \end{array}$$

Solving for S and rounding to the nearest thousandth of a liter, we get:

$$\frac{6S}{6}=\frac{10}{6}$$

$$\frac{\cancel{6}S}{\cancel{6}}=\frac{\cancel{2}\cdot 5}{\cancel{2}\cdot 3}$$

$$S=\frac{5}{3}=1.\overline{6}$$

$$\boxed{S\cong 1.667}$$

Substituting this value of S into the original first equation, we can solve for T:

$$\begin{array}{rcl} S+T &=& 5 \\ 1.667+T &=& 5 \\ 1.667+T &=& 5 \\ -1.667 & & -1.667 \\ \hline \boxed{T} &=& \boxed{3.333} \end{array}$$

You must mix 3.333 liters of the 10% acid solution with 1.667 liters of the 70% acid solution to obtain 5 liters of a 30% acid solution.

For the following Extended Examples, use the method of elimination to solve the linear systems you define.

Extended Example 5a

How many gallons of 30% alcohol solution and 90% alcohol solution must be mixed to end up with exactly 12 gallons of a 50% alcohol solution? (Round your answers to the nearest hundredth of a gallon.)

You have a jar containing 68 coins, consisting entirely of nickels and dimes, worth a total of $5.25.
How many of each type of coin are in the jar?

A solar heating system costs $23,000 to install with a yearly operating cost of $340. An electric heating system costs $5,600, but incurs a yearly operating cost of $3,600. How many (whole) years will pass before the total cost of the solar heating system is less than the total cost of the electric heating system?

How much 2% salt solution must be mixed with how much 8% salt solution to make 50 milliliters of a 7% salt solution? (Round to the nearest thousandth of a milliliter.)

SYSTEMS OF LINEAR INEQUALITIES

Introduction

A **system of linear inequalities** is a linear system with inequalities instead of equations. For example, the pair of linear inequalities,

$$\begin{cases} 2x + 3y \leq 6 \\ 3x - 2y \geq 6, \end{cases}$$

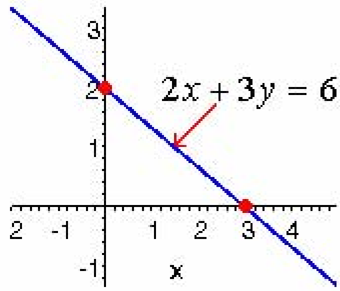
is a system of linear inequalities. The solution set of such a system consists of the points that satisfy all of the inequalities in the system. To graph the solution set of a system, we graph each inequality in the system; the overlap of all the solution regions is the solution set of the system.

SYSTEMS OF LINEAR INEQUALITIES

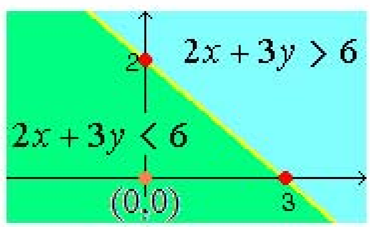
First, let's take a stroll down memory lane...

Review: Graphing a Linear Inequality in Two Variables

Recall that the solution set of an equation, say $2x + 3y = 6$, is a line in the plane. This line goes through the points (3, 0) and (0, 2), its x and y-intercepts:



On one side of the line, it is always true that $2x + 3y > 6$, while on the other side of the line, it is always true that $2x + 3y < 6$.



2 of 21

SYSTEMS OF LINEAR INEQUALITIES

Review, *continued*

There are a few more facts to remember before we get started.

Recall the two-step process for graphing the solution to a linear inequality:

- Step 1 is to replace the inequality symbol with an equal sign and graph that line.
- Step 2 is to decide which side of the line is included in the solution set, and to shade that side.
  - Step 2 amounts to checking a test point that is not on the line to see if it satisfies the inequality. If so, the side of the line including the test point is shaded; if not, the side of the line opposite the test point is shaded. The origin is used whenever possible, since zeros are easy to work with and for the origin, $x = 0$ and $y = 0$.

Graphing Systems of Linear Inequalities

To graph a system of linear inequalities, you simply repeat the same two-step process reviewed above, once for each of the system's inequalities, and put all of the results on the same graph.

The overlap of all the shaded regions that you end up with is the region that satisfies all the inequalities of the system at the same time; this is the solution set of the system of linear inequalities.

Let's look at some examples to help make this clear.

EXAMPLE A

Graph the solution to this system: $\begin{cases} 3x + 2y \le 6 \\ 2x - 3y \ge 6 \end{cases}$.

First, we will use the two-step process for each of these two inequalities.

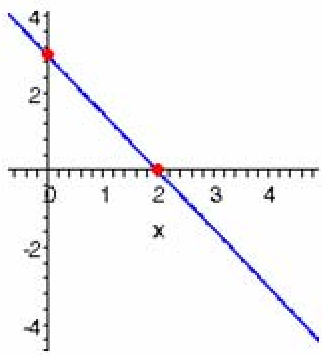
$3x + 2y \le 6$

Step 1: Graph the line $3x + 2y = 6$.

First, find two points on the line. The x and y-intercepts are often easy to find.

$3x + 2y = 6$	x	y	
$3 \cdot 0 + 2 \cdot 3 = 6$	0	3	y-intercept
$3 \cdot 2 + 2 \cdot 0 = 6$	2	0	x-intercept

Plot the points and graph the line. The line is solid since the inequality is "less than or equal to" and therefore the line is included in the solution.



continued...

Example A, continued…

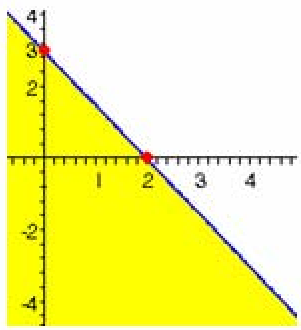
$3x + 2y \le 6$

Step 2: Is test point (0, 0) included in the solution set?

To see if the origin, (0, 0), satisfies the linear inequality, substitute the zeros into the inequality:

$$3x + 2y \le 6$$
$$3 \cdot 0 + 2 \cdot 0 \overset{?}{\le} 6$$
$$0 \overset{?}{\le} 6 \quad \text{...Yes!}$$

So, we shade the side of the line that contains the origin:



continued…

Example A, continued …

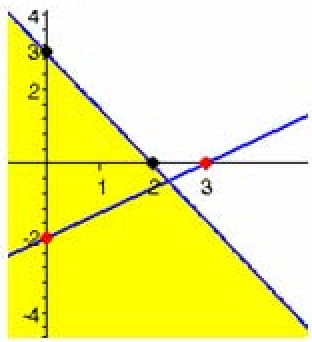
Next, we repeat the two-step process with the other inequality.

$2x - 3y \geq 6$

Step 1: Graph the line $2x - 3y = 6$.
First, find two points on the line (the x and y -intercepts are often easy to find):

$2x - 3y = 6$	x	y	
$2 \cdot 0 - 3 \cdot (-2) = 6$	0	–2	y -intercept
$2 \cdot 3 - 3 \cdot 0 = 6$	3	0	x -intercept

Plot the points and graph the line over the previous graph. The line is solid since the inequality is "greater than or equal to."



continued...

SYSTEMS OF LINEAR INEQUALITIES

Example A, continued…

$\boxed{2x - 3y \geq 6}$

Step 2: Is test point (0, 0) included in the solution set?

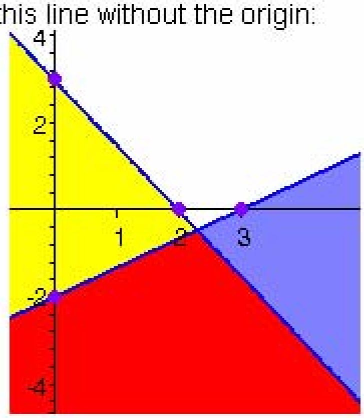
To see if the origin satisfies the linear inequality, substitute the zeros into the inequality:

$$2x - 3y \geq 6$$

$$2 \cdot 0 + 3 \cdot 0 \overset{?}{\geq} 6$$

$$0 \overset{?}{\geq} 6 \quad \ldots\text{No!}$$

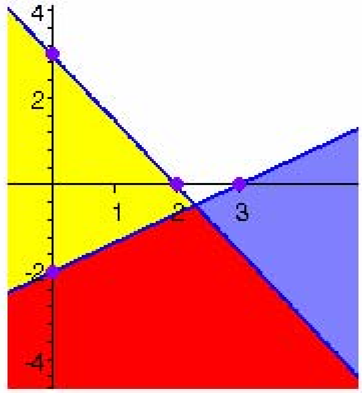
So we shade the side of this line without the origin:



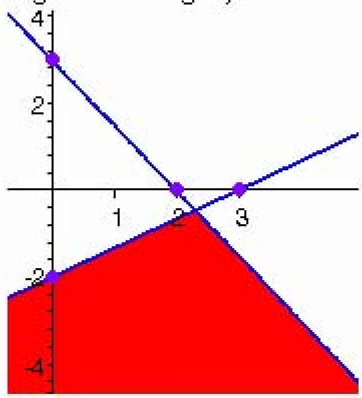
continued…

SYSTEMS OF LINEAR INEQUALITIES

Example A, continued…



The side of that last line that does not contain the origin consists of the red region (below) and violet region (to the right). The overlap of the two inequalities is the lower red region (along with its edges). This is the graph of the solution set of this system:



$$\begin{cases} 3x + 2y \le 6 \\ 2x - 3y \ge 6 \end{cases}$$

You might find this animation of Example A helpful: VIEW ANIMATION

SYSTEMS OF LINEAR INEQUALITIES

Extended Example 1a

Graph the solution to this system: $\begin{cases} y - 3x \leq -6 \\ y \geq -x + 1 \end{cases}$.

Determining Where the Solution Set Lies

There is sometimes an easier way to determine which side of the line to shade. This occurs when all the inequalities are in slope-intercept form (in the sense that if all the inequality symbols were replaced by equal signs, the resulting linear equations would be in slope-intercept form). The simple rule is:

If the inequality is of the form $y > mx + b$, then shade above the line.

If the inequality is of the form $y < mx + b$, then shade below the line.

In other words:

$\boxed{y >} \Rightarrow y$ is larger $\Rightarrow \boxed{\uparrow}$

$\boxed{y <} \Rightarrow y$ is smaller $\Rightarrow \boxed{\downarrow}$

This should make perfect sense. After all, y is the vertical coordinate.

When $y = mx + b$, then the point (x, y) is **on** the line,

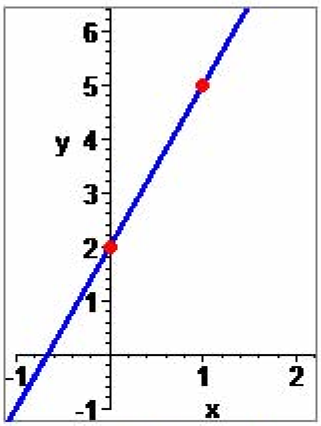
so if $y > mx + b$, then the point (x, y) is **above** the line,

and if $y < mx + b$, then the point (x, y) is **below** the line.

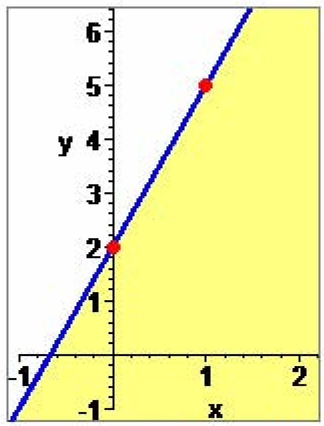
EXAMPLE B

Graph $y \le 3x + 2$.

To graph the inequality $y \le 3x + 2$, we graph the line $y = 3x + 2$ first. The points (0, 2) and (1, 5) are on the line, which is graphed below. The inequality is $\boxed{y \le}$, "y is less than or equal to," and so, we shade below the line, $\boxed{\downarrow}$:



$y = 3x + 2$



$y \le 3x + 2$

SYSTEMS OF LINEAR INEQUALITIES

EXAMPLE C

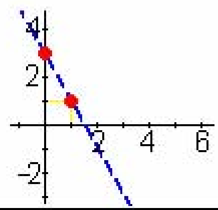
Graph the solution to this system: $\begin{cases} y < 3 - 2x \\ y > \dfrac{3}{5}x - 2 \end{cases}$.

We'll use the familiar two-step process for each of these two inequalities.

$y < 3 - 2x$

Step 1: Graph the line $y = 3 - 2x$. Since this is a strict inequality, <, the line should be dashed rather than solid. The points on the line are not included in the solution region.

First, find two points on the line, plot the points, and graph the line:

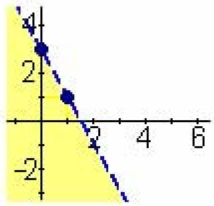
$y = 3 - 2x$	x	y	
$3 = 3 - 2 \cdot 0$	0	3	
$1 = 3 - 2 \cdot 1$	1	1	

continued...

SYSTEMS OF LINEAR INEQUALITIES

Example C, continued...

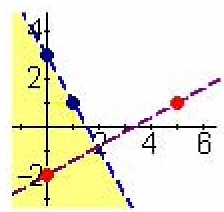
Step 2: Which side contains the solution set? The equation of the line is in slope-intercept form. We can tell which side to shade directly from the inequality. In the inequality $y < 3 - 2x$, "$y <$" means "y is smaller than." The bottom side of the line is shaded:



Next, we repeat the two-step process with the other inequality.

$$y > \frac{3}{5}x - 2$$

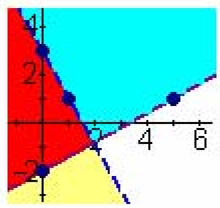
Step 1: Graph the line $y = \frac{3}{5}x - 2$. Since this is a strict inequality, >, the line should be dashed rather than solid. First, find two points on the line, and plot the points. Graph the line over the previous graph:

$y = \frac{3}{5}x - 2$	x	y	
$-2 = \frac{3}{5} \cdot 0 - 2$	0	−2	
$1 = \frac{3}{5} \cdot 5 - 2$	5	1	

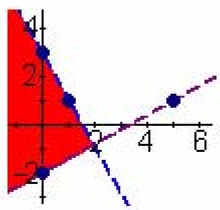
continued...

Example C, continued…

Step 2: Which side contains the solution set? The equation of the line is in slope-intercept form. We can tell which side to shade directly from the inequality. In the inequality, "$y >$" means "y is larger than." The top side of the line is shaded:



The top side of this line consists of the red region (left) and turquoise region (above). The overlap of this inequality with the previous one is the red region (left), which is the graph of the solution set of this system of linear inequalities. The dashed borders are excluded from the solution set:



SYSTEMS OF LINEAR INEQUALITIES

Extended Example 2a

Graph the solution to this system: $\begin{cases} y > 4 - 5x \\ y \le x - 1 \end{cases}$.

SYSTEMS OF LINEAR INEQUALITIES

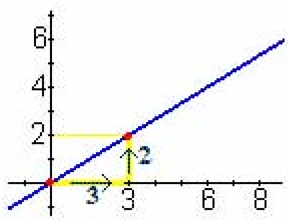
EXAMPLE D

Graph the solution to this system: $\begin{cases} y \geq \frac{2}{3}x \\ y \leq -\frac{3}{4}x + 6 \end{cases}$.

We'll use the two-step process for each of these two inequalities.

$y \geq \frac{2}{3}x$

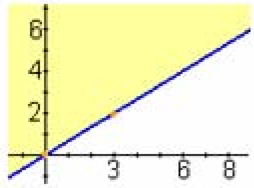
Step 1: Graph the line $y = \frac{2}{3}x$. This line is solid due to the "greater than or equal to" symbol ($\geq$). This equation can be written as $y = \frac{2}{3}x + 0$, so the line has slope $\frac{2}{3}$ and y-intercept $(0,0)$, the origin. Plot $(0,0)$ and use the fact that the slope $= \frac{2}{3} = \frac{\text{rise}}{\text{run}}$. If we run 3 and rise 2, we get back to the line:



continued...

Example D, continued...

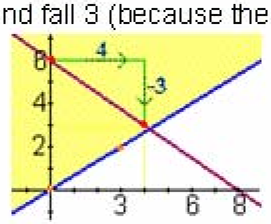
Step 2: Which side is included in the solution set? The equation is in slope intercept form with "$y \geq$", "y greater than or equal to," which means we shade the upper side of the line, where the y-coordinates are above the line:



Next, we repeat the two-step process with the other inequality.

$$y \leq -\frac{3}{4}x + 6$$

Step 1: Graph the line $y = -\frac{3}{4}x + 6$. This line is solid due to the "less than or equal to" symbol ($\leq$). This equation is in slope-intercept form, so the line has slope $-\frac{3}{4}$ and y-intercept $(0,6)$. Plot $(0,6)$ and use the fact that the slope $= -\frac{3}{4} = \frac{-3}{4} = \frac{\text{rise}}{\text{run}}$. If we run 4 and fall 3 (because the rise is negative), we get back to the line:



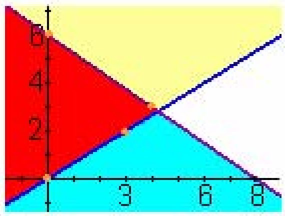
continued...

SYSTEMS OF LINEAR INEQUALITIES

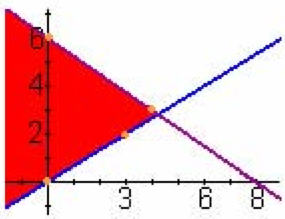
Example D, continued…

Step 2: Which side is included in the solution set? The equation is in slope intercept form with "$y \leq$", "y less than or equal to," which means we shade the <u>lower</u> side of the line, where the y-coordinates are below the line:

This is the red region (left) along with the turquoise region (bottom) in the figure:



The solution region is the red region (left) including its upper and lower borders.



SYSTEMS OF LINEAR INEQUALITIES

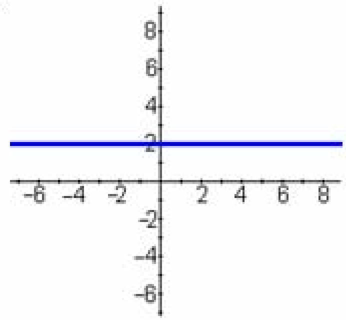
EXAMPLE E

Graph the solution to this system: $\begin{cases} y \le 2 \\ x > 3 \end{cases}$.

We'll use the two-step process for each of these two inequalities.

$y \le 2$

Step 1: Graph the horizontal line $y = 2$. This line is solid due to the "less than or equal to" sign ($\le$).



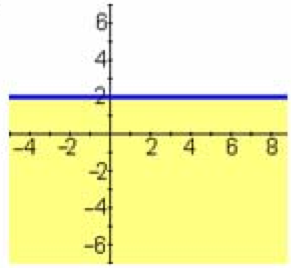
Step 2: Which side is included in the solution? We'll use the origin, $(0, 0)$, for our test point. Substituting this point into the original inequality, we get:

$$(0, 0)$$
$$\downarrow$$
$$y \le 2$$
$$0 \le 2 \quad \ldots\text{Yes!}$$

continued...

Example E, continued...

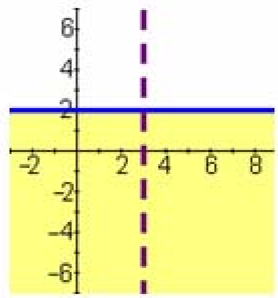
The side containing the origin is shaded:



Next, we repeat the two-step process with the other inequality.

$x > 3$

Step 1: Graph the vertical line $x = 3$. Since this is a strict inequality, $>$, the line should be dashed rather than solid. The points on the line are not included in the solution region.



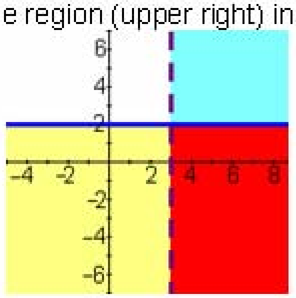
continued...

Example E, continued...

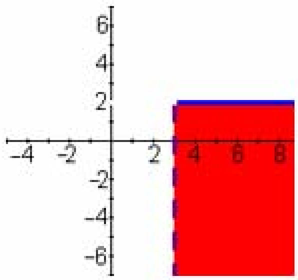
Step 2: Which side contains the solutions? We'll use the origin, $(0,0)$, for our test point. Substituting this point into the original inequality, we get:

$$(0, 0)$$
$$\downarrow$$
$$x > 3$$
$$0 > 3 \quad \ldots\text{No!}$$

The side of this line without the origin is shaded. This is the red region (lower right) along with the turquoise region (upper right) in the figure:



The solution region is the red region (lower right) including its upper border, but not including its left border.



END OF LESSON

Graph the solution set of the system of linear inequalities.

$$\begin{cases} 2x + y \le 2 \\ x - 2y \ge 4 \end{cases}$$

Graph the solution set of the system of linear inequalities.

$$\begin{cases} y > \dfrac{5}{7}x - 2 \\ y > -\dfrac{5}{4}x + 2 \end{cases}$$

Graph the solution set of the system of linear inequalities.

$$\begin{cases} y > 3 \\ x \leq 1 \end{cases}$$