KINDS OF NUMBERS

Introduction

This lesson briefly reviews the real numbers and introduces variables.

KINDS OF NUMBERS

Digits

Digits are the ten number symbols used to write any number. These are the digits:

0 1 2 3 4 5 6 7 8 9

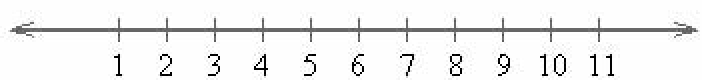
Counting Numbers

Counting numbers are the numbers we use when we count, and the rules of place value allow us to go beyond nine. They are also called "natural numbers." The counting numbers are

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 . . .

Note:

- The . . . after the number 12 in the list above indicates that the counting numbers continue beyond 12, even though we've stopped listing them.



Note:

- Counting numbers do not include zero. The number line is usually drawn with arrows on both ends, but the counting numbers begin at 1 and do not include zero or anything to the left of 1.

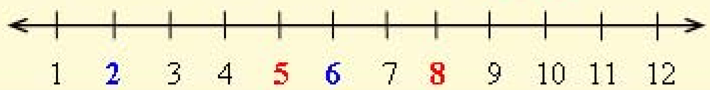
KINDS OF NUMBERS

Comparing Counting Numbers

Any number to the right of another number on the number line is said to be **greater than** the other number.

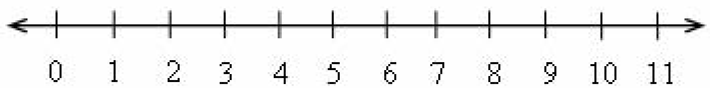
Any number to the left of another number on the number line is said to be **less than** the other number.

Examples: We say "6 is greater than 2" and we write $6 > 2$.
We can write "5 is less than 8" as $5 < 8$.



Whole Numbers

Whole numbers include the counting numbers, but they begin with zero.

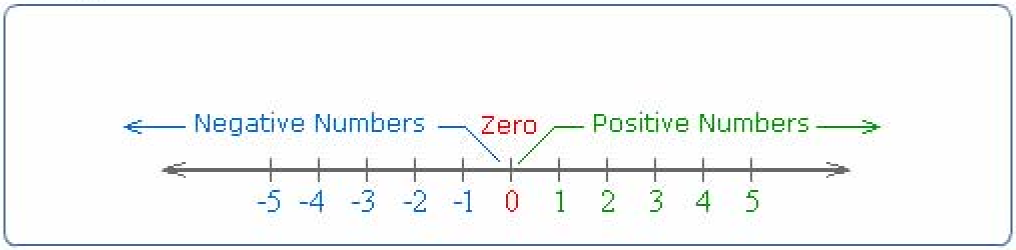


The concepts "greater than" and "less than" also apply to whole numbers.

KINDS OF NUMBERS

Integers

Whole numbers together with negative numbers are called **integers**. On a number line, negative numbers are to the left of zero, and positive numbers are to the right. Numbers that are the same distance from zero but in opposite directions are called **opposite numbers**. For example, the numbers 3 and −3 are opposites.



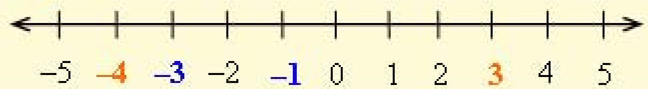
Note:

- The negative sign is always used with the negative numbers.
- No sign is used with positive numbers.
- Zero is neither positive nor negative.

Comparing Integers

The concepts greater than and less than also apply to integers, but be careful with the sign!

Examples: $-1 > -3$ because −1 is to the right of −3 on the number line.
$-4 < 3$ because −4 is to the left of 3 on the number line.



KINDS OF NUMBERS

Rational Numbers

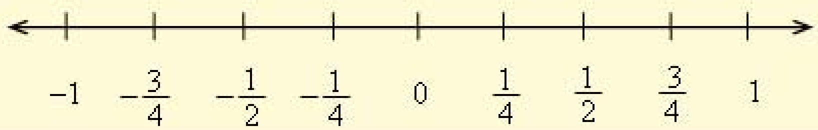
The set of **rational numbers** includes any number that can be written as a **fraction**.

Examples:

The number -3 is a rational number because it can be written as $\frac{-3}{1}$, which is a fraction. So, the set of rational numbers includes all integers.

The number $\frac{2}{7}$ is another rational number.

The number line below shows a few rational numbers.



The concepts greater than and less than also apply to fractions.

Examples: $\frac{1}{2} < \frac{3}{4}$

$-\frac{1}{2} > -\frac{3}{4}$

Improper Fractions

A fraction is called **improper** if the integer in the numerator is greater than the integer in the denominator. Students are often taught to convert improper fractions into **mixed number** notation, which consists of a whole number with a fraction to its right.

Example: The improper fraction $\frac{8}{3}$ is equal to the mixed number $2\frac{2}{3}$.

Mixed Numbers

In algebra we rarely use mixed number notation. Improper fractions are preferred. The algebraic expression ab means $a \times b$, but the mixed number $2\frac{2}{3}$ actually means $2+\frac{2}{3}$. We avoid this notational discrepancy by simply avoiding mixed numbers. When confronted by a mixed number in an algebra class, the first step is almost always to convert it to an improper fraction.

For example, we can convert the mixed number $7\frac{3}{5}$ to an improper fraction using the following steps.

$$7\frac{3}{5}=7+\frac{3}{5}=\frac{7}{1}\left(\frac{5}{5}\right)+\frac{3}{5}=\frac{7\times5+3}{5}=\frac{35+3}{5}=\frac{38}{5}$$

KINDS OF NUMBERS

Fractions in Decimal Form

Fractions may be written in decimal form. To convert a fraction into decimal form, we divide the numerator by the denominator using long division or a calculator.

Often we round the decimal that results. Note that rounding a decimal number results in an approximation to the fraction, rather than the true, exact value of the fraction.

When decimals have a repeating set of digits, we can indicate the repetition by either "… " or by a bar over the repeating digits. (Note that the three dots are also used to indicate an infinite set of numbers that goes on without ending, even if there is no repetition.)

Examples:

$\frac{1}{2} = 0.5$	The decimal form is the exact value of the fraction.
$\frac{1}{3} \cong 0.33$	The decimal form is only an approximation of the fraction's exact value.
$\frac{1}{3} = 0.333\ldots$	The decimal form is the exact value of the fraction. The "… " indicates endlessly repeating 3's.
$\frac{1}{3} = 0.\overline{3}$	This decimal notation is neater than the version above The bar indicates that the 3s endlessly repeat.

KINDS OF NUMBERS

Irrational Numbers

Rational numbers can be written as terminating or repeating decimals. **Irrational numbers** have nonterminating, nonrepeating decimal expansions.

Together, rational and irrational numbers make up the set of **real numbers**. A straight line is a model of the set of real numbers; each point on the line corresponds to a unique real number.

Examples of real numbers follow.

$\pi = 3.1415926535...$	An irrational number
5 or $\frac{5}{1}$	A rational number
$\frac{3}{4}$ or 0.75	A rational number
$\sqrt{2} = 1.414213562...$	An irrational number
$\frac{5}{7}$ or $0.\overline{714285}$	A rational number

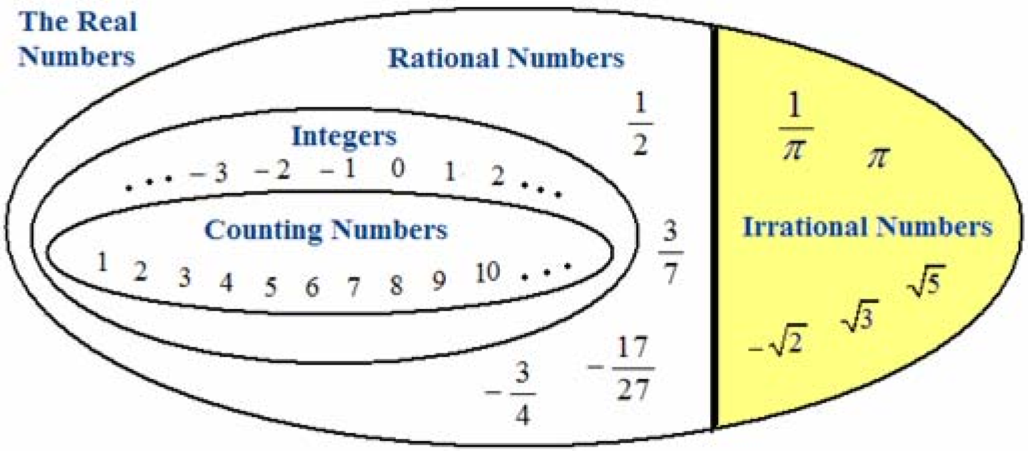
Note:

- In the examples $\pi = 3.1415926535...$ and $\sqrt{2} = 1.414213562...$, the dots indicate that the decimal expansions continue forever, although in these cases not by the repetition of any simple patterns.

KINDS OF NUMBERS

Real Numbers

The diagram below depicts the kinds of numbers that make up the set of real numbers, along with representatives of each type of number.



This diagram shows that all counting numbers are integers, and all integers are rational numbers, but that no irrational numbers are rational. All of these numbers are real numbers, which consist of the rational and irrational numbers combined.

KINDS OF NUMBERS

Using Variables to Represent Numbers

In algebra, we use letters and other symbols to represent numbers. Letters that represent numbers in algebra are called **variables.**

The rules that govern how numbers behave also govern how variables behave in algebra. In fact, we'll see that the easiest way to describe the various properties of numbers is by using variables, the "common nouns" of algebra.

Note:

- You cannot take for granted what kind of number a variable may represent.

Example: If the variable y represents any real number, that number could be positive, negative, zero, a whole number, a fraction, or an irrational number.

Example: If the variable x represents a whole number, we know that x cannot be a fraction, and it cannot be negative.

KINDS OF NUMBERS

Introduction

This lesson briefly reviews the real numbers and introduces variables.

KINDS OF NUMBERS

Digits

Digits are the ten number symbols used to write any number. These are the digits:

0 1 2 3 4 5 6 7 8 9

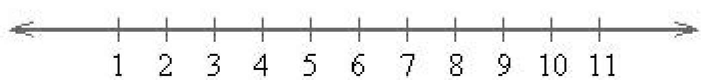
Counting Numbers

Counting numbers are the numbers we use when we count, and the rules of place value allow us to go beyond nine. They are also called "natural numbers." The counting numbers are

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 . . .

Note:

- The . . . after the number 12 in the list above indicates that the counting numbers continue beyond 12 , even though we've stopped listing them.



Note:

- Counting numbers do not include zero. The number line is usually drawn with arrows on both ends, but the counting numbers begin at 1 and do not include zero or anything to the left of 1 .

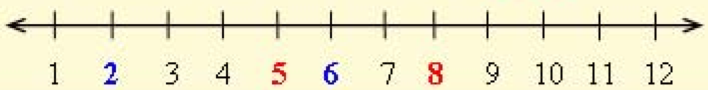
KINDS OF NUMBERS

Comparing Counting Numbers

Any number to the right of another number on the number line is said to be **greater than** the other number.

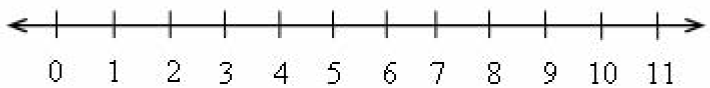
Any number to the left of another number on the number line is said to be **less than** the other number.

Examples: We say "6 is greater than 2" and we write $6 > 2$.
We can write "5 is less than 8" as $5 < 8$.



Whole Numbers

Whole numbers include the counting numbers, but they begin with zero.

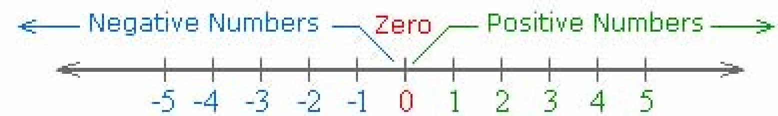


The concepts "greater than" and "less than" also apply to whole numbers.

KINDS OF NUMBERS

Integers

Whole numbers together with negative numbers are called **integers**. On a number line, negative numbers are to the left of zero, and positive numbers are to the right. Numbers that are the same distance from zero but in opposite directions are called **opposite numbers**. For example, the numbers 3 and -3 are opposites.



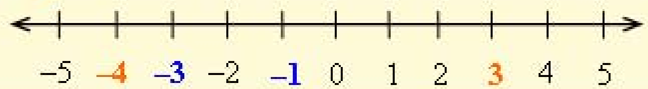
Note:

- The negative sign is always used with the negative numbers.
- No sign is used with positive numbers.
- Zero is neither positive nor negative.

Comparing Integers

The concepts greater than and less than also apply to integers, but be careful with the sign!

Examples: $-1 > -3$ because -1 is to the right of -3 on the number line.
$-4 < 3$ because -4 is to the left of 3 on the number line.



Rational Numbers

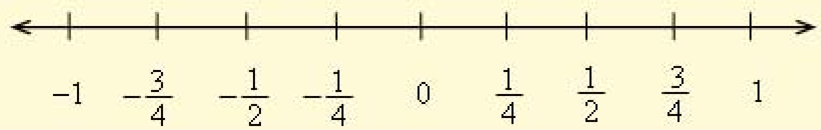
The set of **rational numbers** includes any number that can be written as a **fraction**.

Examples:

The number -3 is a rational number because it can be written as $\frac{-3}{1}$, which is a fraction. So, the set of rational numbers includes all integers.

The number $\frac{2}{7}$ is another rational number.

The number line below shows a few rational numbers.



The concepts greater than and less than also apply to fractions.

Examples: $\frac{1}{2} < \frac{3}{4}$

$-\frac{1}{2} > -\frac{3}{4}$

KINDS OF NUMBERS

Improper Fractions

A fraction is called **improper** if the integer in the numerator is greater than the integer in the denominator. Students are often taught to convert improper fractions into **mixed number** notation, which consists of a whole number with a fraction to its right.

Example: The improper fraction $\frac{8}{3}$ is equal to the mixed number $2\frac{2}{3}$.

Mixed Numbers

In algebra we rarely use mixed number notation. Improper fractions are preferred. The algebraic expression ab means $a \times b$, but the mixed number $2\frac{2}{3}$ actually means $2+\frac{2}{3}$. We avoid this notational discrepancy by simply avoiding mixed numbers. When confronted by a mixed number in an algebra class, the first step is almost always to convert it to an improper fraction.

For example, we can convert the mixed number $7\frac{3}{5}$ to an improper fraction using the following steps.

$$7\frac{3}{5}=7+\frac{3}{5}=\frac{7}{1}\left(\frac{5}{5}\right)+\frac{3}{5}=\frac{7\times 5+3}{5}=\frac{35+3}{5}=\frac{38}{5}$$

Fractions in Decimal Form

Fractions may be written in decimal form. To convert a fraction into decimal form, we divide the numerator by the denominator using long division or a calculator.

Often we round the decimal that results. Note that rounding a decimal number results in an approximation to the fraction, rather than the true, exact value of the fraction.

When decimals have a repeating set of digits, we can indicate the repetition by either "..." or by a bar over the repeating digits. (Note that the three dots are also used to indicate an infinite set of numbers that goes on without ending, even if there is no repetition.)

Examples:

$\frac{1}{2} = 0.5$	The decimal form is the exact value of the fraction.
$\frac{1}{3} \cong 0.33$	The decimal form is only an approximation of the fraction's exact value.
$\frac{1}{3} = 0.333\ldots$	The decimal form is the exact value of the fraction. The "..." indicates endlessly repeating 3's.
$\frac{1}{3} = 0.\overline{3}$	This decimal notation is neater than the version above The bar indicates that the 3s endlessly repeat.

KINDS OF NUMBERS

Irrational Numbers

Rational numbers can be written as terminating or repeating decimals. **Irrational numbers** have nonterminating, nonrepeating decimal expansions.

Together, rational and irrational numbers make up the set of **real numbers**. A straight line is a model of the set of real numbers; each point on the line corresponds to a unique real number.

Examples of real numbers follow.

$\pi = 3.1415926535...$	An irrational number
5 or $\frac{5}{1}$	A rational number
$\frac{3}{4}$ or 0.75	A rational number
$\sqrt{2} = 1.414213562...$	An irrational number
$\frac{5}{7}$ or $0.\overline{714285}$	A rational number

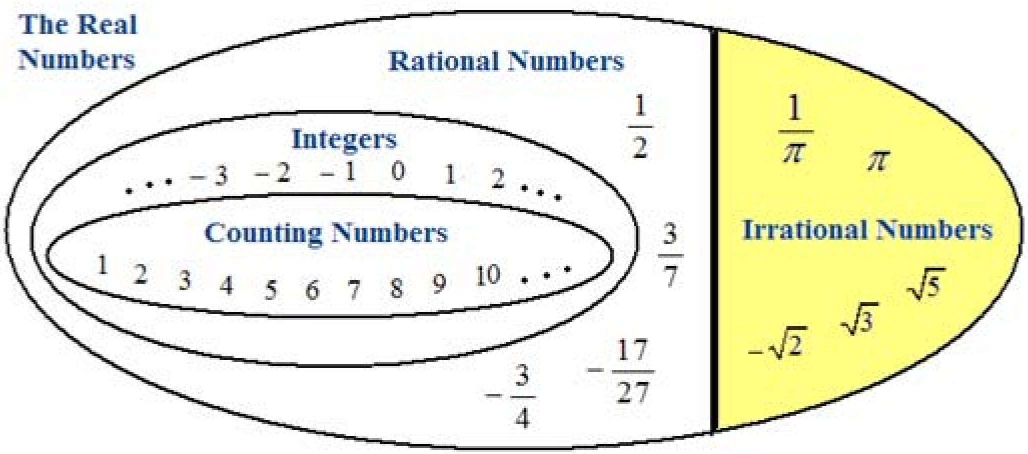
Note:

- In the examples $\pi = 3.1415926535...$ and $\sqrt{2} = 1.414213562...$, the dots indicate that the decimal expansions continue forever, although in these cases not by the repetition of any simple patterns.

KINDS OF NUMBERS

Real Numbers

The diagram below depicts the kinds of numbers that make up the set of real numbers, along with representatives of each type of number.



This diagram shows that all counting numbers are integers, and all integers are rational numbers, but that no irrational numbers are rational. All of these numbers are real numbers, which consist of the rational and irrational numbers combined.

Using Variables to Represent Numbers

In algebra, we use letters and other symbols to represent numbers. Letters that represent numbers in algebra are called **variables.**

The rules that govern how numbers behave also govern how variables behave in algebra. In fact, we'll see that the easiest way to describe the various properties of numbers is by using variables, the "common nouns" of algebra.

Note:

- You cannot take for granted what kind of number a variable may represent.

Example: If the variable y represents any real number, that number could be positive, negative, zero, a whole number, a fraction, or an irrational number.

Example: If the variable x represents a whole number, we know that x cannot be a fraction, and it cannot be negative.

BEHAVIOR OF NUMBERS AND VARIABLES

Introduction

The rules that we use to perform operations on numbers also apply to operations involving variables. In this lesson, we review these rules. These rules, or "properties," of the real numbers can be thought of as the rules of the game of algebra.

1 of 7

Commutative Property of Addition

Numbers or variables can be added in any order without affecting their sum. This rule is the **commutative property of addition**. We can use the commutative property to rewrite expressions.

Examples: $2+3=\boxed{5}$ and $3+2=\boxed{5}$

$a+b=b+a$

Even when we add several numbers or variables in a row, the order of the numbers or variables being added isn't important.

Examples: $2+5+1+9=\boxed{17}$ and $5+9+1+2=\boxed{17}$

$a+b+c+d=c+b+d+a$

Note that the sum $a+a$ can be written as $2a$, and that $5x$ means $x+x+x+x+x$.

Example: $3c+4c=7c$

The left-hand side has 3 c's plus 4 more c's, which equals 7 c's in total.

Subtraction is Not Commutative

Numbers or variables cannot be subtracted in any order. Their difference will be affected. Subtraction is not commutative.

Examples: $5-2=3$ but $2-5=-3$

$a-b\neq b-a$

BEHAVIOR OF NUMBERS AND VARIABLES

Commutative Property of Multiplication

Numbers or variables can be multiplied in any order without affecting their product. This rule is the **commutative property of multiplication.** We can use the commutative property of multiplication to rewrite expressions.

Examples: $2 \times 8 = \boxed{16}$ and $8 \times 2 = \boxed{16}$

$3 \times 9 \times 2 \times 10 = 9 \times 3 \times 10 \times 2$

$a \times b \times c = c \times b \times a$

Note:

- $a \times b \times c$ is usually written as abc, so we can say $abc = cba$.

Division is Not Commutative

Numbers or variables cannot be divided in any order. Their quotient will be affected. Division is not commutative.

Examples: $\frac{4}{5} = 0.8$ but $\frac{5}{4} = 1.25$

$\frac{3}{8} \neq \frac{8}{3}$

$\frac{a}{b} \neq \frac{b}{a}$

Note:

- The fractions above are called **reciprocals** of each other.

$\frac{a}{b}$ is the reciprocal of $\frac{b}{a}$

BEHAVIOR OF NUMBERS AND VARIABLES

Associative Property of Addition

When adding numbers or variables, we can group them in any way. The **associative property of addition** states that how you group the numbers being added will not affect their sum.

Examples:

$$2+(3+5)+6=2+8+6=\boxed{16} \quad \text{and} \quad (2+3)+(5+6)=5+11=\boxed{16}$$

$$(a+b)+(c+d)=a+(b+c)+d=a+b+c+d$$

Associative Property of Multiplication

When multiplying a series of numbers or variables, we can group them in any way. This rule is the **associative property of multiplication**.

Examples:

$$3\times(5\times 2)=\boxed{30} \quad \text{and} \quad (3\times 5)\times 2=\boxed{30}$$

$$(4\times 3)\times 7=4\times(3\times 7)$$

$$a(bc)=(ab)c$$

BEHAVIOR OF NUMBERS AND VARIABLES

Distributive Property

Consider solving $3\times(5+2)$. You can take two approaches:

1. Add the numbers in parentheses, and then multiply the sum by 3.
   So: $3\times 7=21$

2. "Distribute" the first factor, 3, through the sum $(5+2)$. In other words, compute 3×5, then 3×2, and then add the two products:
   $$3\times(5+2)=(3\times 5)+(3\times 2)=15+6,$$
   which equals 21.

Now consider $a\times(b+c)$. Can we use the first approach from above?
Because we don't know the values for a, b, and c, we cannot simplify $b+c$.

We can use the second approach, however:

$$a\times(b+c)=(a\times b)+(a\times c)=ab+ac.$$

continued...

Distributive Property, *continued*

Multiplying a by b and then by c is an example of the **distributive property**. This property states that for all numbers a, b, and c, it is always true that

$$a(b+c)=ab+ac.$$

Examples:

$$2(w+z)=2w+2z$$

$$d(h+g)=dh+dg$$

Note that this property also applies when the variables are subtracted:

$$a\times(b-c)=ab-ac.$$

Examples:

$$b(s-v)=bs-bv$$

$$3(4-7)=3\times4-3\times7$$

BEHAVIOR OF NUMBERS AND VARIABLES

Properties for 0 and 1

0 and 1 have some unique properties:

$2+0=2$ $a+0=a$	Adding zero doesn't change the number or variable. For any real number a, it's true that $a+0=a$. We say that 0 is the **additive identity** for the real numbers. The number 0 is unique in this respect.
$2-0=2$ $a-0=a$	Subtracting zero also has no effect.
$2\times0=0$ $a\times0=0$	Multiplying by zero equals zero.
$2/0$ is undefined $a/0$ is undefined	A number cannot be divided by zero.
$0/2=0$ $0/a=0$ for $a\neq0$	Zero divided by any number other than zero is zero.
$2\times1=2$ $a\times1=a$	Multiplying by 1 doesn't change the number or variable. For any real number a, it's true that $a\times1=a$. We say that 1 is the **multiplicative identity**. The number 1 is unique in this respect.
$2/1=2$ $a/1=a$	Dividing by 1 doesn't change the number or variable.

END OF LESSON

MULTIPLYING AND DIVIDING SIGNED NUMBERS AND VARIABLES

Introduction

It is important to memorize the rules for multiplying and dividing signed numbers and variables.

Rules for Multiplication

Rule 1: Multiplying Numbers with the Same Sign
The product of two numbers with the same sign is positive.

Examples: $2 \times 7 = 14$

$-2 \times (-7) = 14$

Rule 2: Multiplying Numbers with Different Signs
The product of two numbers with different signs is negative.

Examples: $2 \times (-7) = -14$

$-2 \times 7 = -14$

Rule 3: Multiplying an Even Number of Negative Numbers
Multiplying an even number of negative numbers gives a positive product.

Example: $-2 \times (-4) \times (-3) \times (-1) \times 2 =$ (Four negative numbers)

$8 \times (-3) \times (-1) \times 2 =$

$-24 \times (-1) \times 2 =$

$24 \times 2 = 48$

Rule 4: Multiplying an Odd Number of Negative Numbers
Multiplying an odd number of negative numbers gives a negative product.

Example: $-2 \times (-4) \times (-3) \times 1 \times 2 =$ (Three negative numbers)

$$8 \times (-3) \times 1 \times 2 =$$

$$-24 \times 1 \times 2 =$$

$$-24 \times 2 = -48$$

Extended Example 1

$(-3)(-6)(-3) = ?$

Rules for Division

The rules that apply to multiplying signed numbers also apply to dividing signed numbers.

Rule 5: Dividing Numbers with the Same Sign
When you divide two numbers with the same sign, the quotient is positive.

Example: $18 \div 3 = 6$

$-18 \div (-3) = 6$

Rule 6: Dividing Numbers with Different Signs
When you divide two numbers with different signs, the quotient is negative.

Example: $18 \div (-3) = -6$

$-18 \div 3 = -6$

MULTIPLYING AND DIVIDING SIGNED NUMBERS AND VARIABLES

Multiplying and Dividing with Variables

The rules for multiplying and dividing signed numbers also apply to variables. Note that it is important to remember that a variable representing a number does not indicate the number's sign. For example, $a(-b) = -ab$, but this product, $-ab$, could be positive, negative, or zero, depending upon whether a and b represent positive or negative numbers.

In algebra, we can indicate multiplication in a variety of ways:

$a \times b$	$a \cdot b$	ab
$(a)(b)$	$a(b)$	$(a)b$

Note:

- $3 \times 5 = 15$ or $3 \cdot 5 = 15$ or $3(5) = 15$
- $4a = 4 \times a$ but $35 \neq 3 \times 5$

We know that $ab = ba$. The order of multiplied variables does not affect the product. But, it is standard practice to respect the order of the alphabet in most cases. You would rarely see the product ba in an answer. However, you might see it in the intermediate steps of solving an equation.

To simplify expressions with both numbers and variables, we handle them separately, starting with the numbers. The variables just "come along for the ride," so to speak.

For example, solve:	$(-4t) \cdot (3s)$
First, multiply the numbers:	$-4(3) = -12$
Now, include the variables:	$-4t(3s) = -12ts = -12st$

Extended Example 2

$(-2s)(-8t) = ?$

Simplify: $(-4x)\times(-8a)$

Simplify: $-4(a-b)$

Simplify each of the following.

$-4(a+(-a))=$

Simplify: $\left(-\frac{4}{2}\right)\times(-a)$

Simplify: $(-a)\left(\frac{b}{c}\right)$

Does $\frac{5-4}{5-3-2}$ have meaning?

OPERATIONS WITH FRACTIONS AND VARIABLES

Introduction

This lesson focuses on multiplying, dividing, adding, and subtracting fractions.

1 of 11

OPERATIONS WITH FRACTIONS AND VARIABLES

Multiplying Fractions

To multiply fractions, multiply the numerators together and multiply the denominators together. The denominators don't have to be the same. Use the same rules you use to multiply signed numbers to multiply signed fractions .

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

Note: a , b , c , and d represent any numbers, except that neither b nor d can be zero.

Examples: $1/2 \times 1/4 = 1/8$

$-6/7 \times \left(-3/4\right) = 18/28 = 9/14$ *Note:* we simplified the answer.

Remember to **simplify** an answer, or write it in **lowest terms**, by canceling any factors common to both the numerator and the denominator. In fact, since you have to cancel common factors anyway, it's better to do so *before* you multiply.

Example: Consider the product $36/11 \cdot 33/36$. If you don't cancel common factors before multiplying, you'll have to write $1188/396$ in lowest terms, since $\frac{36}{11} \cdot \frac{33}{36} = \frac{36 \cdot 33}{11 \cdot 36} = \frac{1188}{396}$. Notice that it's easier to start by canceling common factors rather than by multiplying:

$$\frac{36}{11} \cdot \frac{33}{36} = \frac{36 \cdot 33}{11 \cdot 36} = \frac{\cancel{36} \cdot 3 \cdot 11}{11 \cdot \cancel{36}} = \frac{3 \cdot 11}{11} = \frac{3 \cdot \cancel{11}}{\cancel{11}} = 3 .$$

Extended Example 1

Multiply: $\frac{16}{111} \cdot \frac{777}{64}$.

Extended Example 2

Multiply: $\frac{-5a}{7} \cdot \frac{3}{4b}$.

Dividing Fractions

To divide fractions, multiply the first fraction by the reciprocal of the second fraction (in other words, flip the second fraction's numerator and denominator).

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc} \quad \text{or} \quad \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

Note: Again, no denominator can be zero.

Examples:

$$\frac{1}{3} \div \frac{2}{5} = \frac{1}{3} \times \frac{5}{2}$$ Change $\div$ to $\times$ and use the second fraction's reciprocal.

$$= \frac{5}{6}$$ Multiply the fractions.

$$\frac{-3}{4} \div \frac{6}{-7} = \frac{-3}{4} \times \frac{-7}{6}$$ Change $\div$ to $\times$ and use the second fraction's reciprocal.

$$= \frac{21}{24} = \frac{7}{8}$$ Multiply the fractions and simplify the answer.

Negative Fractions

Note that negative fractions can be written in several ways:

$$\frac{a}{-b} = \frac{-a}{b} = -\frac{a}{b}$$

OPERATIONS WITH FRACTIONS AND VARIABLES

Extended Example 3

Divide: $\dfrac{\dfrac{-3}{5a}}{\dfrac{7b}{4}}$.

6 of 11

Adding and Subtracting Fractions with Like Denominators

If fractions have the same denominators (also called **like denominators**), you can add or subtract the numerators. The denominator stays the same.

$$\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b} \quad \text{and} \quad \frac{a}{b}-\frac{c}{b}=\frac{a-c}{b}$$

Study the following examples involving fractions with the *same denominators*:

$\frac{1}{3}+\frac{1}{3}=\frac{1+1}{3}=\frac{2}{3}$		$\frac{a}{b}+\frac{a}{b}=\frac{2a}{b}$
$\frac{-1}{3}+\frac{-1}{3}=\frac{-1+(-1)}{3}=\frac{-2}{3}$		$\frac{-a}{b}+\frac{-a}{b}=-\frac{2a}{b}$
$\frac{1}{3}+\left(-\frac{1}{3}\right)=\frac{1+(-1)}{3}=\frac{0}{3}=0$		$\frac{a}{b}+\frac{-a}{b}=\frac{0}{b}=0$
$\frac{3}{4}-\left(\frac{1}{4}\right)=\frac{3-1}{4}=\frac{2}{4}=\frac{1}{2}$		$\frac{3a}{b}-\left(\frac{a}{b}\right)=\frac{2a}{b}$
$\frac{1}{5}-\left(\frac{-2}{5}\right)=\frac{1-(-2)}{5}=\frac{3}{5}$		$\frac{a}{b}-\left(-\frac{2a}{b}\right)=\frac{3a}{b}$

OPERATIONS WITH FRACTIONS AND VARIABLES

Adding and Subtracting Fractions with Unlike Denominators

To add or subtract fractions with denominators that are not the same, we must first find a **common denominator**. Then, we can rewrite the fractions using that common denominator and add them just as we would add any fractions with like denominators. This is shown in the following examples. Pay attention to the signs as you review the examples.

EXAMPLE A

$$\frac{1}{3}+\frac{1}{2}=?$$

These fractions have unlike denominators. We need to find a common denominator in order to add them together. We know that 3 and 2 both divide into 6 evenly, so 6 is a common multiple of 3 and 2; in other words, 6 is a common denominator of $1/3$ and $1/2$. First, we need to rewrite $1/3$ with a 6 in the denominator, so we have to multiply the 3 in the denominator by 2.

Therefore we will multiply $\frac{1}{3}$ by $\frac{2}{2}$: $\quad \frac{1}{3}\times\frac{2}{2}=\frac{5}{6}$.

Notice that we multiplied $1/3$ by $2/2$, which is 1, the multiplicative identity, so we did not change the value of $1/3$. Next, we need to rewrite $1/2$ with a 6 in the denominator. We'll need to multiply its denominator by 3 to get 6, so we'll multiply the fraction by the multiplicative identity $3/3$, which won't change the fraction's value. The steps are combined here:

$$\frac{1}{3}+\frac{1}{2}=\left(\frac{1}{3}\times\frac{2}{2}\right)+\left(\frac{1}{2}\times\frac{3}{3}\right)=\frac{2}{6}+\frac{3}{6}=\frac{2+3}{6}=\frac{5}{6}$$

OPERATIONS WITH FRACTIONS AND VARIABLES

EXAMPLE B

$$-\frac{1}{6}+\left(-\frac{5}{12}\right)=?$$

$$-\frac{1}{6}+\left(-\frac{5}{12}\right)=\left(-\frac{1}{6}\right)\times\frac{2}{2}+\left(-\frac{5}{12}\right)$$ The common denominator is 12.

$$=\left(-\frac{2}{12}\right)+\left(-\frac{5}{12}\right)$$

$$=\frac{-2+(-5)}{12}=\frac{-7}{12}$$, which we can write as $-\frac{7}{12}$.

EXAMPLE C

$$\frac{3}{10}-\frac{4}{25}=?$$

$$\frac{3}{10}-\frac{4}{25}=\frac{3}{10}\times\frac{5}{5}-\left(\frac{4}{25}\times\frac{2}{2}\right)$$ The common denominator is 50.

$$=\frac{15}{50}-\frac{8}{50}=\frac{7}{50}$$

9 of 11

OPERATIONS WITH FRACTIONS AND VARIABLES

EXAMPLE D

$$\frac{11}{40}+\left(\frac{-1}{8}\right)=?$$

$$\frac{11}{40}+\left(\frac{-1}{8}\right)=\frac{11}{40}+\left(-\frac{1}{8}\right)\left(\frac{5}{5}\right)$$ The common denominator is 40.

$$=\frac{11}{40}+\left(\frac{-5}{40}\right)$$

$$=\frac{11+(-5)}{40}$$

$$=\frac{6}{40}=\frac{3}{20}$$ Simplify to find the final answer.

EXAMPLE E

$$\frac{-4}{15}+\frac{-3}{20}=?$$

$$\frac{-4}{15}+\frac{-3}{20}=\frac{-4}{15}\left(\frac{4}{4}\right)+-\frac{3}{20}\left(\frac{3}{3}\right)$$ The common denominator is 60.

$$=\frac{-16}{60}+\frac{-9}{60}$$

$$=\frac{-16+-9}{60}$$

$$=-\frac{25}{60}=-\frac{5}{12}$$ Simplify to find the final answer.

OPERATIONS WITH FRACTIONS AND VARIABLES

Extended Example 4

Add: $\frac{3}{7}+\frac{2}{5}$.

END OF LESSON

Which is greater, $-\frac{1}{3}$ **or** $-\frac{1}{4}$?

Simplify: $\left(\frac{2}{5}\right)\left(\frac{-3}{-4}\right)$

Simplify: $\dfrac{\frac{4}{-5}}{\frac{2}{3}}$

Simplify each of the following.

$\frac{-3}{5} \div \frac{2}{7} =$

Simplify: $\frac{3}{4} - \frac{5}{6}$

Simplify: $\frac{4}{b} - \frac{7}{a}$

Simplify: $\frac{3}{8} + \frac{5}{22}$

Simplify each of the following.

$\frac{b}{15} - \frac{c}{9} =$

SQUARES, SQUARE ROOTS, AND ABSOLUTE VALUE

Introduction

This lesson explains squared numbers, the square root, and the idea of absolute value.

SQUARES, SQUARE ROOTS, AND ABSOLUTE VALUE

Squared Numbers

To **square** a number means to multiply the number by itself.

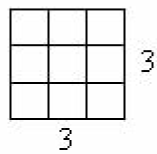
The superscript ^{2} after a number indicates that the number is to be squared. For example, 3^2 means "square the number 3," or "3 squared" for short.

Variables may be squared. For any real number, a:

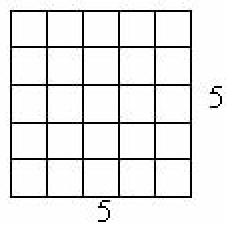
$$a^2 = a \times a$$

It might help to think of the area of a square.

Square the number 3: $3^2 = 3 \times 3 = 9$



Square the number 5: $5^2 = 5 \times 5 = 25$



SQUARES, SQUARE ROOTS, AND ABSOLUTE VALUE

Not all numerical squares can easily be pictured as geometric squares.

Examples:

Square $\frac{1}{2}$: $\left(\frac{1}{2}\right)^2 = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4}$

Note: When we square this fraction, the product is less than the number being squared.

Square (-5): $(-5)^2 = (-5)(-5) = 25$ Here, -5 is squared.

However, note:

$-5^2 = -(5^2) = -(5 \times 5) = -25$ Here, only 5 is squared.

Parentheses must be used if the negative sign is to be included with the number being squared. Why? It's always true that $-a = (-1)a$, so

$$-5^2 = (-1)5^2 = (-1)25 = -25.$$

Extended Example 1

Square $\left(-\frac{3}{4}\right)$

The Square Root

Finding the square root of a number means finding a factor that, when multiplied by itself, equals that number.

Example: A square root of 16 is 4 because $4 \times 4 = 16$. The other square root of 16 is -4 because $-4 \times (-4) = 16$.

Each positive number has two square roots, one positive and the other negative. For example $(-5)^2 = (-5)(-5) = 25$, and also $5^2 = 5 \cdot 5 = 25$, so both 5 and -5 are square roots of 25. The positive square root is called the **principal root**. The $\sqrt{\ }$ symbol, called the **radical** or the **square root sign**, indicates the positive square root of the number inside the radical. The expression under the radical sign is called the **radicand**. Numbers such as $1, 4, 9, 16, 25, \ldots$ are called **perfect squares** because they are the squares of integers.

Examples:

$\sqrt{9} = 3$ because $3 \times 3 = 9$

$\sqrt{144} = 12$ because $12 \cdot 12 = 144$

$\sqrt{\frac{16}{25}} = \frac{4}{5}$ because $\frac{4}{5} \times \frac{4}{5} = \frac{16}{25}$

$\sqrt{0.09} = 0.3$ because $0.3 \times 0.3 = 0.09$

To indicate a negative root, you must put a negative sign in front of the radical.

Examples: $-\sqrt{16} = -4$ $\quad -\sqrt{\frac{9}{64}} = -\frac{3}{8}$

Question: Evaluate $\sqrt{121}$. Explain your answer.

Question: Find the principle square root of 81 .

Extended Example 2

Find all the square roots of 169 .

The Square Root of a Negative Number

The square root of a negative real number cannot be any real number.

To see why this statement is true, let's imagine that $\sqrt{-4} = R$. If so, then $R \times R = -4$. But that's impossible for any real number R, because if R is positive, then $R \times R$ is positive, and so it can't equal -4 ! And, if R is negative, then $R \times R$ is still positive since a negative times a negative is a positive. Either way, it doesn't equal -4.

If you try to make your calculator take the square root of a negative number, it will probably display something like "ERROR."

Estimating the Square Root of a Non-Perfect Square

Consider the radical $\sqrt{8}$. Since 8 isn't a perfect square, we can't calculate this square root easily. But we do know that 8 is between the perfect squares 4 and 9. This means that the square root of 8 will be between the square root of 4 and the square root of 9. In other words,

$$\sqrt{4} < \sqrt{8} < \sqrt{9} .$$

Taking the square roots of the radicals on each end, we get

$$2 < \sqrt{8} < 3;$$

Therefore, $\sqrt{8}$ is some number between 2 and 3.

continued...

SQUARES, SQUARE ROOTS, AND ABSOLUTE VALUE

Estimating the Square Root of a Non-Perfect Square, *continued*

You can get better estimates by systematically guessing—we know $\sqrt{8}$ is between 2 and 3. The mid-point between 2 and 3 is 2.5. Squaring 2.5, we get $2.5^2 = 6.25$. Since this number is still smaller than 8, we now know that $\sqrt{8}$ is between 2.5 and 3. We can repeat this process, restricting $\sqrt{8}$ to smaller and smaller intervals until we know $\sqrt{8}$ quite accurately. But, if your calculator has a square root key, then accurate approximations to square roots are just a button push away! My calculator says $\sqrt{8} \cong 2.8284271212475$. The more digits you use, the more accurate your calculations will be, so it's generally best not to round off until the very last step of a calculation.

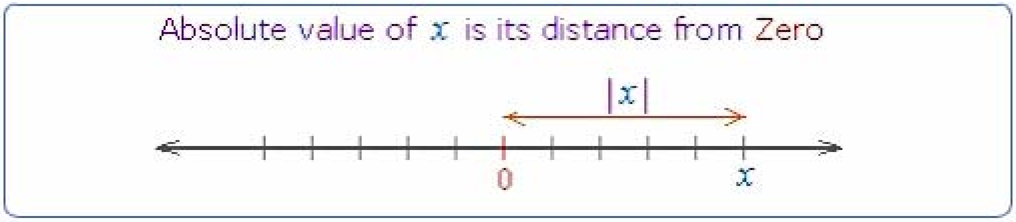
Extended Example 3

$\sqrt{57}$ is between which two consecutive positive integers? Also, use your calculator to find $\sqrt{57}$ to 5 decimal places and to 3 decimal places.

SQUARES, SQUARE ROOTS, AND ABSOLUTE VALUE

Absolute Value

The **absolute value** of a number is its distance from 0 on the number line. The absolute value of x is written $|x|$. Absolute values are never negative because they refer to only the magnitude, or size, of a number, not its sign.



Note that $|ab| = |a| \times |b|$. But $|a+b| \neq |a| + |b|$.

We can illustrate this by letting $a = 3$ and $b = -7$:

$|a+b| = |3+(-7)| = |3-7| = |-4| = 4$ but $|a|+|b| = |3|+|-7| = 3+7 = 10$

$$\text{So, } |3+(-7)| \neq |3| + |-7|$$

Question: Find the absolute value of -39.

SQUARES, SQUARE ROOTS, AND ABSOLUTE VALUE

The Square Root of a Variable

You can have variables under a radical too, as long at the they result in a positive quantity under the radical sign.

You now know that $\sqrt{9} = 3$, but there's more juice to be squeezed from this fruit! Look at that expression in another way: $\sqrt{9} = \sqrt{3 \cdot 3} = \sqrt{3^2} = 3$. Also, $\sqrt{4^2} = 4$, $\sqrt{5^2} = 5$, $\sqrt{\pi^2} = \pi$, and so on. The square root of the square of any non-negative number equals the number itself.

> If a represents any non-negative real number, then $\sqrt{a^2} = a$.

The property above is *not* true when a represents a negative number. For example, suppose that $a = -2$. Then $\sqrt{a^2} = \sqrt{(-2)^2} = \sqrt{4} = 2$, which is not equal to a. When the negative number was squared, the negative was lost. But remember that absolute values also result in negative signs being lost. Indeed, if a is a negative number then $\sqrt{a^2} = |a|$. This formula also works if a is positive, since the absolute value of a positive number is that positive number itself.

> If a represents any real number, then $\sqrt{a^2} = |a|$.

END OF LESSON

Find the square root: $\sqrt{324}$

Find the square root: $\sqrt{\frac{49}{64}}$

Find the square root: $\sqrt{0.0036}$

$\sqrt{230}$ is between what two consecutive whole numbers?

Find the square root (variables represent non-negative quantities): $\sqrt{a^2b^2c^2}$

Between what two consecutive whole numbers is the square root of the number?

$\sqrt{193}$

$\sqrt{925}$ is between what two consecutive whole numbers?

ORDER OF OPERATIONS

Introduction

Without rules to follow, we might end up with several answers for the same problem. To avoid this, we follow a set of rules called the **order of operations**. This lesson reviews this set of rules.

ORDER OF OPERATIONS

Simplify:

$12 + 3 \times 2$

If we add first, we get:

$15 \times 2 = 30$

If we multiply first, we get:

$12 + 6 = 18$

Which one is correct?

Rule 1: Always multiply and divide (from left to right) before adding and subtracting (from left to right).

Rule 1 lets us solve the equation above in only one way—by multiplying first:

$$12 + 3 \times 2 = 12 + 6 = 18$$

Additional examples:

Simplify: $24 \div 4 + 4 \times 2$.

$$\begin{aligned} 24 \div 4 + 4 \times 2 &= 6 + 4 \times 2 \\ &= 6 + 8 \\ &= 14 \end{aligned}$$

Simplify: $3 \times 4 + 16 \div 2 - 3$.

$$\begin{aligned} 3 \times 4 + 16 \div 2 - 3 &= 12 + 16 \div 2 - 3 \\ &= 12 + 8 - 3 \\ &= 20 - 3 \\ &= 17 \end{aligned}$$

ORDER OF OPERATIONS

Simplify:
$40 \div 5 \times 4$

If we divide first, we get:
$8 \times 4 = 32$

If we multiply first, we get:
$40 \div 20 = 2$

Which one is correct?

Rule 1 from the previous screen requires that multiplication or division be performed <u>as encountered from left to right</u>. Division is the first operation we should perform, so the answer is 32:

$$40 \div 5 \times 4 = 8 \times 4 = 32.$$

Example: Simplify $100 \div 25 \times 4 + 6(-4) - 19$.

$100 \div 25 \times 4 + 6(-4) - 19 = 4 \times 4 + 6(-4) - 19$ Divide $100 \div 25 = 4$

$= 16 + 6(-4) - 19$ Multiply $4 \times 4 = 16$

$= 16 - 24 - 19$ Then, multiply $6(-4) = -24$

$= -8 - 19$ Then, subtract $16 - 24 = -8$

$= -27$ Subtract.

ORDER OF OPERATIONS

Grouping Symbols

Grouping symbols include parentheses, ( ), brackets, [ ], or braces, { }. There really is no difference between one type of grouping symbol and another. Just remember that a left grouping symbol, such as (, is always paired with a right grouping symbol—in this case,). Sometimes various grouping symbols are used in the same problem to help you see which left grouping symbol goes with which right grouping symbol.

Two additional symbols that function as grouping symbols are the radical sign, $\sqrt{}$, and the fraction bar: — . You've encountered both symbols already, but you might not have thought of them as grouping symbols.

Grouping symbols are sometimes used only to eliminate ambiguity. They are usually used, however, to force expressions to be evaluated in a way that would otherwise violate the order of operation conventions.

Examples:

$$3 \times 5 + 8 = 15 + 8 = 23$$

but

$$3 \times (5 + 8) = 3 \times 13 = 39$$

$$5 + 2 \times 10 - 4 = 5 + 20 - 4 = 25 - 4 = 21$$

but

$$5 + 2 \times (10 - 4) = 5 + 2 \times 6 = 5 + 12 = 17$$

ORDER OF OPERATIONS

The rules for the order of operations specify when to evaluate expressions within grouping symbols.

Rule 2: Perform operations within parentheses and other grouping symbols first. Then follow Rule 1.

Example A

Simplify $\frac{15-9}{3}$.

Simplify within the grouping symbol (the fraction bar) first:

$$\frac{15-9}{3} = \frac{6}{3}$$

Then, divide. $= 6 \div 3$

$= 2$

The fraction bar separates the numerator from the denominator. The numerator and denominator have to be simplified before we can divide.

Example B

Simplify $5+4\sqrt{17+83}$.

Simplify within the grouping symbol, the radical sign, first.

$$5+4\sqrt{17+83} = 5+4\sqrt{100}$$ *Note:* $\sqrt{100} = 10$ because $10 \cdot 10 = 100$

$= 5+4(10)$

Then, multiply. $= 5+40$

$= 45$

5 of 7

ORDER OF OPERATIONS

When layers of grouping symbols are nested within each other, it helps to have a rule to guide us.

> **Rule 3:** When an expression contains more than one grouping symbol, perform the operations within the innermost grouping symbol first. Then work toward the outermost grouping symbol.

Example C

Simplify $2\left[4(3+2)-5\right]$.

$$2\left[4(3+2)-5\right] = 2\left[4(5)-5\right]$$ Simplify within innermost grouping symbol first.

$$= 2\left[20-5\right]$$ Then, simplify within the brackets.

$$= 2 \cdot 15$$
$$= 30$$

Note: Brackets, [and], are sometimes used as outer grouping symbols when parentheses are nested inside them.

Example D

Simplify $40\left[18-(15-48)-12\right]$.

$$40\left[18-(15-48)-12\right] = 40\left[18-(-33)-12\right]$$
$$= 40\left[18+33-12\right]$$
$$= 40\left[51-12\right]$$
$$= 40 \cdot 39$$
$$= 1{,}560$$

ORDER OF OPERATIONS

Extended Example 1a

Simplify $(2+5)\times 6 \div 3+3$.

END OF LESSON

Use the rules for the order of operations to simplify:

$14 + 7 \times 6 \div 2 + 8$

Use the rules for the order of operations to simplify:

$(-52) \div (-26) \div 2 \times (-13)$

Use the rules for the order of operations to simplify:

$3(-3) - 2\left[4 + \left(5 - 4(-3)\right)\right] - 17$

Use the rules for the order of operations to simplify:

$83d - \left[87 + 3(18d - 46) + 5\right] - 12$

Use the rules for the order of operations to simplify:

$$-\left[18n+12\left(\frac{4n-8n}{16}\right)+5m\right]$$

Use the rules for the order of operations to simplify:

$$-\left[18n+12\left(\frac{4n-8n}{16}\right)+5m\right]$$