THE SQUARE ROOT PROPERTY

Introduction

In the last chapter, you saw how squaring both sides of the equal sign was useful in solving equations involving square roots. In this lesson, you will see that taking the square root of both sides of the equation is a useful and powerful technique for solving equations involving squared expressions. Earlier, you learned how to solve equations by factoring. Unfortunately, many quadratic equations cannot be factored.

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THE SQUARE ROOT PROPERTY

The Square Root Property
If $A^2 = B$, then $A = \sqrt{B}$ or $A = -\sqrt{B}$.

This property gives us an easy way to solve many quadratic equations. Consider the equation:

$$x^2 = 4.$$

Previously you learned to first subtract 4 from both sides, to get

$$x^2 - 4 = 0.$$

And then factor the quadratic equation. This quadratic equation is the difference of two squares:

$$x^2 - 2^2 = 0$$

$$(x+2) \cdot (x-2) = 0$$

$$\downarrow \qquad \downarrow$$

$$\boxed{x = -2} \text{ or } \boxed{x = 2}$$

Using the square root property is <u>much</u> easier (as you'll see in Example A)!

Note:

- When a problem has two opposite solutions, instead of writing $x = -2$ or $x = 2$, you can use the shorthand $\boxed{x = \pm 2}$. This is read as "x equals plus or minus 2."

Using this "plus or minus" sign, the square root property reads:

If $A^2 = B$, then $A = \pm\sqrt{B}$.

THE SQUARE ROOT PROPERTY

EXAMPLE A

Solve $x^2 = 4$, using the square root property.

Applying the square root property, we get:

$$x^2 = 4$$
$$x = \pm\sqrt{4}$$
$$x = \pm 2$$

That's all there is to it!

EXAMPLE B

Solve: $x^2 = 17$.

Applying the square root property, we get:

$$x^2 = 17$$
$$x = \pm\sqrt{17}$$

This last example could not have been solved using the usual factoring method. However, it was quite easy to solve using the square root property.

THE SQUARE ROOT PROPERTY

Extended Example 1a

Solve: $x^2 = 200$.

THE SQUARE ROOT PROPERTY

EXAMPLE C

Solve: $x^2 = -7$.

This equation has no real solutions. But don't worry, i comes to the rescue! Applying the square root property, we get:

$$x^2 = -7$$
$$= \pm\sqrt{-7}$$
$$x = \pm i\sqrt{7}$$

We found two imaginary solutions. Nonetheless, these are valid solutions.

Note:

- You may sometimes see the complex number i placed after the radical instead of before it. It means the same thing either way it's written, since multiplication is commutative: $AB = BA$, or $i\sqrt{7} = \sqrt{7}\,i$.

Question: Solve: $x^2 = -100$.

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Question: Solve: $x^2 = -68$.

Question: Solve: $x^2 = -224$.

THE SQUARE ROOT PROPERTY

EXAMPLE D

Solve: $(3x-5)^2 = 7$.

Applying the square root property, we get:

$$(3x-5)^2 = 7$$
$$3x-5 = \pm\sqrt{7}$$

Now we continue solving for x:

$$\begin{aligned} 3x-5 &= \pm\sqrt{7} \\ +5 \quad & \quad +5 \\ \hline 3x &= 5 \pm \sqrt{7} \\ \frac{\cancel{3}x}{\cancel{3}} &= \frac{5 \pm \sqrt{7}}{3} \\ x &= \frac{5 \pm \sqrt{7}}{3} \end{aligned}$$

Remember, when you replace the $\pm$ with a $+$, you get one solution; when you replace the $\pm$ with a $-$, you get another solution.

THE SQUARE ROOT PROPERTY

Extended Example 2a

Solve: $(x+9)^2 = 27$.

THE SQUARE ROOT PROPERTY

EXAMPLE E

Solve: $(4x+3)^2-12=0$.

Get the squared expression by itself on one side of the equal sign:

$$\begin{array}{r} (4x+3)^2-12 \;=\; 0 \\ +12 \quad +12 \\ \hline (4x+3)^2 = 12 \end{array}$$

Now we can use the square root property:

$$(4x+3)^2 = 12$$

$$4x+3=\pm\sqrt{12}$$

Simplifying the radical, the equation becomes:

$$4x+3=\pm\sqrt{12}=\pm\sqrt{4\cdot 3}=\pm\sqrt{4}\cdot\sqrt{3}$$

$$4x+3=\pm 2\sqrt{3}$$

Continuing to solve for x :

$$\begin{array}{r} 4x+3=\pm 2\sqrt{3} \\ -3 \quad -3 \\ \hline 4x=-3\pm 2\sqrt{3} \end{array}$$

$$\frac{\cancel{4}x}{\cancel{4}}=\frac{-3\pm 2\sqrt{3}}{4}$$

$$x=\frac{-3\pm 2\sqrt{3}}{4}$$

THE SQUARE ROOT PROPERTY

Extended Example 3a

Solve: $(3x+7)^2 - 99 = 0$.

THE SQUARE ROOT PROPERTY

EXAMPLE F

Solve: $(2x-13)^2=-1$.

Applying the square root property, we get:

$$(2x-13)^2=-1$$
$$2x-13=\pm\sqrt{-1}$$

Recall that $i=\sqrt{-1}$. We can replace the entire radical with i and solve for x:

$$2x-13=\pm i$$
$$\underline{+13\quad +13}$$
$$2x=13\pm i$$
$$\frac{\cancel{2}x}{\cancel{2}}=\frac{13\pm i}{2}$$
$$\boxed{x=\frac{13\pm i}{2}} \text{ or } \boxed{x=\frac{13}{2}\pm\frac{1}{2}i}$$

The last form of the answer gives the real and imaginary parts of the solutions clearly. Notice that the two complex solutions are conjugates:

$$\frac{13}{2}+\frac{1}{2}i \text{ and } \frac{13}{2}-\frac{1}{2}i.$$

THE SQUARE ROOT PROPERTY

Extended Example 4a

Solve: $(x+5)^2 + 25 = 0$.

EXAMPLE G

Solve: $(3x-7)^2 = 2$. Give decimal approximations to your solutions, rounded to the nearest thousandth.

Applying the square root property, we get:

$$(3x-7)^2 = 2$$
$$3x-7 = \pm\sqrt{2}$$

Continuing to solve for x:

$$\begin{aligned} 3x-7 &= \pm\sqrt{2} \\ +7 \quad & \quad +7 \\ \hline 3x &= 7 \pm \sqrt{2} \\ \frac{\not{3}x}{\not{3}} &= \frac{7 \pm \sqrt{2}}{3} \\ x &= \frac{7 \pm \sqrt{2}}{3} \end{aligned}$$

continued...

THE SQUARE ROOT PROPERTY

Example G, continued...

Lastly, we need to evaluate these solutions. You will need a calculator. Start by calculating the square root of 2. Use all the precision given by the calculator.

Never round any calculation until the very end. Rounding is the very last thing you should do:

$$\sqrt{2} \cong 1.41421356237 .$$

Now we can calculate each of our solutions:

$$x = \frac{7+\sqrt{2}}{3} \cong \frac{7+1.41421356237}{3} = \frac{8.41421356237}{3} \cong 2.80473785412$$

$$x = \frac{7-\sqrt{2}}{3} \cong \frac{7-1.41421356237}{3} = \frac{5.58578643763}{3} \cong 1.86192881254$$

Rounded to the nearest thousandth, our solutions are:

$$x \cong 2.805 \text{ and } x \cong 1.862 .$$

Extended Example 5a

Solve: $(x+1)^2 = 3$. Give decimal approximations to your solutions. Round to the nearest hundred-thousandth.

Solve.

$x^2 = 98$

Rewrite in the form $ax^2 + bx + c = 0$, where $a \neq 0$:

$$\frac{2x}{x+1} = \frac{3}{x-2}$$

Solve by factoring: $x^2 + 17x + 42 = 0$

Solve.

$$\left(18x+29\right)^2=-18$$

Solve by factoring: $\dfrac{1}{x-2}+\dfrac{1}{2x+6}=\dfrac{1}{4}$

Solve the equation and give decimal approximations to your solutions, rounded to the nearest ten-thousandth.

$$\left(10x+11\right)^2=24$$

COMPLETING THE SQUARE

Introduction

In this lesson, you will learn how to "complete the square." Armed with this technique, you will be able to solve any quadratic equation. Completing the square is the key step in deriving the famous quadratic formula, which you will learn in the next section. This powerful and useful method is as simple as dividing by two and then squaring.

Completing the Square: Overview

The process of **completing the square** is the process of finding a constant that must be added to

$$x^2 + Bx$$

such that the result is the square of a binomial. In other words, we must fill in the blanks:

$$x^2 + Bx + \boxed{?} = \left(x + \boxed{?}\right)^2.$$

Recall the binomial square formula,

$$a^2 + 2ab + b^2 = (a + b)^2.$$

Letting $a = x$ yields:

$$x^2 + 2xb + b^2 = (x + b)^2$$

$$x^2 + 2bx + b^2 = (x + b)^2$$

The last formula holds the secret of completing the square. The coefficient of x on the left, $2b$, is twice the constant b that occurs in the parentheses on the right. In other words, b is one-half of $2b$.

COMPLETING THE SQUARE

Going back to our fill-in-the-blanks problem,

$$x^2 + Bx + \boxed{?} = \left(x + \boxed{?}\right)^2,$$

We see that the constant in the parentheses on the right, $\boxed{?}$, is half of B:

$$x^2 + Bx + \boxed{?} = \left(x + \boxed{\frac{B}{2}}\right)^2.$$

Then the constant $\boxed{?}$ is the square of $\frac{B}{2}$. In other words:

÷ 2

$$x^2 + Bx + \boxed{\left(\frac{B}{2}\right)^2} = \left(x + \boxed{\frac{B}{2}}\right)^2$$

square

This may all seem rather complicated, but the process consists of simply taking half of B and then squaring that quotient. That's all there is to it: dividing by two and then squaring!

COMPLETING THE SQUARE

EXAMPLE A

Complete the square by filling in the blanks: $x^2 + 8x + \boxed{?} = \left(x + \boxed{?}\right)^2$.

We need to take half of 8 and then square that result:

$$x^2 + 8x + \boxed{16} = \left(x + \boxed{4}\right)^2$$

÷ 2

square

Therefore, we have: $x^2 + 8x + 16 = \left(x + 4\right)^2$.

EXAMPLE B

Complete the square by filling in the blanks: $x^2 - 5x + \boxed{?} = \left(x - \boxed{?}\right)^2$.

Notice that the x term is negative. So, the binomial is a difference rather than a sum. We need to take half of 5 and then square that result

$$x^2 - 5x + \boxed{\frac{25}{4}} = \left(x - \boxed{\frac{5}{2}}\right)^2$$

÷ 2

square

Therefore, we have: $x^2 - 5x + \frac{25}{4} = \left(x - \frac{5}{2}\right)^2$.

COMPLETING THE SQUARE

Question: Complete the square by filling in the blanks: $x^2 + 12x + \boxed{?} = \left(x + \boxed{?}\right)^2$.

Question: Complete the square by filling in the blanks: $x^2 - 20x + \boxed{?} = \left(x - \boxed{?}\right)^2$.

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COMPLETING THE SQUARE

Question: Complete the square by filling in the blanks: $x^2 - 7x + \boxed{?} = \left(x - \boxed{?}\right)^2$.

COMPLETING THE SQUARE

EXAMPLE C

Solve: $x^2 + 14x = 10$.

We need to add something to both sides of the equation that makes the left side a perfect square. Set up the problem for completing the square. Divide the coefficient of x by 2, square it, and add it to both sides of the equation:

$$x^2 + 14x + \boxed{49} = 10 + \boxed{49}$$

÷ 2 ↘ ↑ **square**

$$\left(x + \boxed{7}\right)^2 = 59$$

The above shows that if we add 49 to both sides of the equal sign, the result is a perfect square on the left side:

$$x^2 + 14x = 10$$

$$x^2 + 14x + 49 = 10 + 49$$

$$(x+7)^2 = 59$$

Now we can apply the square root property:

$$x + 7 = \pm\sqrt{59}$$

Subtract 7 from both sides, and we solve for x:

$$\begin{aligned} x + 7 &= \pm\sqrt{59} \\ -7 \quad & \quad -7 \\ \hline x &= -7 \pm \sqrt{59} \end{aligned}$$

COMPLETING THE SQUARE

Extended Example 1a

Solve: $x^2 + 18x = 12$.

COMPLETING THE SQUARE

EXAMPLE D

Solve: $x^2+6x-11=0$.

Start by adding 11 to both sides. Then the equation will look just like the ones featured in the previous examples and can be solved in the same way:

$$\begin{array}{rcl} x^2+6x-11 &=& 0 \\ +11 && +11 \\ \hline x^2+6x &=& 11 \end{array}$$

We need to add something to both sides of the equation that makes the left side a perfect square.

$$x^2 + 6x + \boxed{9} = 11 + \boxed{9}$$

$\div 2$ ↘ ↑ **square**

$$\left(x + \boxed{3}\right)^2 = 20$$

Now we apply the square root property and simplify the resulting radical:

$$(x+3)^2 = 20$$

$$x+3 = \pm\sqrt{20} = \pm\sqrt{4\cdot 5} = \pm\sqrt{4}\cdot\sqrt{5}$$

$$x+3 = \pm 2\sqrt{5}$$

Subtract 3 from both sides and solve for x:

$$\begin{array}{rcl} x+3 &=& \pm 2\sqrt{5} \\ -3 && -3 \\ \hline x &=& -3\pm 2\sqrt{5} \end{array}$$

COMPLETING THE SQUARE

Extended Example 2a

Solve: $x^2 - 16x - 20 = 0$.

COMPLETING THE SQUARE

Sometimes a quadratic equation has no real solutions but instead has two complex solutions.

Extended Example 3a

Solve: $x^2+2x+3=0$.

COMPLETING THE SQUARE

Whenever the coefficient of the x^2 term is something other than 1, the first step is to divide both sides of the equation by that coefficient. (Recall that the term x^2 has coefficient 1 since $x^2 = 1x^2$.)

For example, to solve $5x^2 + 30x - 1 = 0$, you start by dividing each coefficient by 5. As you can see, this amounts to dividing each term by 5:

$$\frac{5x^2 + 30x - 1}{5} = \frac{0}{5}$$

$$\frac{5}{5}x^2 + \frac{30}{5}x - \frac{1}{5} = 0$$

$$\frac{\cancel{5}}{\cancel{5}}x^2 + \frac{\cancel{5} \cdot 6}{\cancel{5}}x - \frac{1}{5} = 0$$

$$x^2 + 6x - \frac{1}{5} = 0$$

Then proceed to solve by completing the square.

COMPLETING THE SQUARE

Extended Example 4a

Solve: $5x^2 + 30x - 1 = 0$.

END OF LESSON

Complete the square, then factor as a perfect square: $x^2 + 10x$

Solve by completing the square.

$x^2 + 12x - 9 = 0$

Solve by completing the square: $2b^2 - 5b - 10 = 0$

Solve by completing the square.

$2x^2 + 2x + 7 = 0$

The length of a rectangular lot is 4 meters more than twice its width. The area of the lot is 96 square meters. What are the dimensions?

Solve by completing the square.

$2y^2 - 6y - 3 = 0$

THE QUADRATIC FORMULA

Introduction

In this lesson, we discover the **quadratic formula** and show how to use it to solve quadratic equations. The quadratic formula allows you to go from a quadratic equation to its solution without having to complete the square.

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THE QUADRATIC FORMULA

Discovering the Quadratic Formula

Finding the quadratic formula is done in precisely the same way as you solved equations in the last section, by completing the square. We start with a quadratic equation in standard form:

$$ax^2 + bx + c = 0.$$

Assume that $a > 0$; the constants b and c may be <u>any</u> real numbers.

Now we will solve this equation for x, just as we did in the last lesson. Only now we are solving <u>all</u> possible quadratic equations. And that is a lot of equations (infinitely many!). The result is called **the quadratic formula**.

We start by dividing both sides of the equation by a:

$$\frac{ax^2 + bx + c}{a} = \frac{0}{a}$$

$$\frac{a}{a}x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$\frac{\cancel{a}}{\cancel{a}}x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

continued...

THE QUADRATIC FORMULA

continued...

We subtract $\frac{c}{a}$ from both sides of the equation:

$$
\begin{array}{rcl}
x^2 + \frac{b}{a}x + \frac{c}{a} & = & 0 \\
-\frac{c}{a} & & -\frac{c}{a} \\
\hline
x^2 + \frac{b}{a}x & = & -\frac{c}{a}
\end{array}
$$

We complete the square:

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$x^2 + \frac{b}{a}x + \boxed{\frac{b^2}{4a^2}} = -\frac{c}{a} + \boxed{\frac{b^2}{4a^2}}$$

$\div 2$ ↘ ↑ **square**

$$\left(x + \boxed{\frac{b}{2a}}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$

continued...

THE QUADRATIC FORMULA

continued…

Find a common denominator and add the rational expressions to the right of the equal sign:

$$\left(x+\frac{b}{2a}\right)^2=-\frac{c\cdot 4a}{a\cdot 4a}+\frac{b^2}{4a^2}=\frac{-4ac}{4a^2}+\frac{b^2}{4a^2}=\frac{-4ac+b^2}{4a^2}$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

Now we can apply the square root property:

$$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}=\frac{\pm\sqrt{b^2-4ac}}{\sqrt{4a^2}}=\frac{\pm\sqrt{b^2-4ac}}{2a}$$

$$x+\frac{b}{2a}=\frac{\pm\sqrt{b^2-4ac}}{2a}$$

Subtracting $\frac{b}{2a}$ from both sides of the equation completes our derivation:

$$x+\frac{b}{2a}-\frac{b}{2a}=\frac{\pm\sqrt{b^2-4ac}}{2a}-\frac{b}{2a}$$

$$x=\frac{\pm\sqrt{b^2-4ac}}{2a}-\frac{b}{2a}=\frac{\pm\sqrt{b^2-4ac}-b}{2a}$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

The Quadratic Formula

The solution to the quadratic equation $ax^2+bx+c=0$, where a is positive, is given by the quadratic formula: **The Quadratic Formula**

$$ax^2+bx+c=0 \xrightarrow[a>0]{\text{assuming}} x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

A Word to the Wise

Memorize this formula!

EXAMPLE A

Solve: $2x^2-6x-5=0$.

To use the quadratic formula, you must identify a, b, and c first:

$$2x^2 \quad -6x \quad -5 \quad = \quad 0$$

$a=2$ $b=-6$ $c=-5$

Substitute these values into the quadratic formula:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-(-6)\pm\sqrt{(-6)^2-4\cdot 2\cdot(-5)}}{2\cdot 2}$$

$$=\frac{6\pm\sqrt{36+40}}{4}$$

$$=\frac{6\pm\sqrt{76}}{4}$$

continued...

Example A, continued…

If you need decimal approximations to the solutions, calculate them from the expression above. Otherwise, we must finish the job by simplifying the radical:

$$x = \frac{6 \pm \sqrt{76}}{4} = \frac{6 \pm \sqrt{2 \cdot 38}}{4} = \frac{6 \pm \sqrt{2 \cdot 2 \cdot 19}}{4} = \frac{6 \pm \sqrt{4 \cdot 19}}{4} = \frac{6 \pm \sqrt{4} \cdot \sqrt{19}}{4}$$

$$x = \frac{6 \pm 2\sqrt{19}}{4}$$

Lastly, we can factor a 2 out of the numerator and cancel it

$$x = \frac{6 \pm 2\sqrt{19}}{4} = \frac{2 \cdot 3 \pm 2 \cdot \sqrt{19}}{2 \cdot 2} = \frac{2 \cdot \left(3 \pm \sqrt{19}\right)}{2 \cdot 2} = \frac{\cancel{2}\left(3 \pm \sqrt{19}\right)}{\cancel{2} \cdot 2}$$

$$x = \frac{3 \pm \sqrt{19}}{2}$$

We can get the solution after substituting the values in the formula.

Extended Example 1a

Solve: $2x^2 + 3x - 3 = 0$.

THE QUADRATIC FORMULA

EXAMPLE B

Solve: $3x^2 - 3x + 7 = 0$.

To use the quadratic formula, you must identify a , b , and c first:

$$3x^2 \quad -3x \quad + \quad 7 \quad = \quad 0$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\boxed{a = 3} \; \boxed{b = -3} \; \boxed{c = 7}$$

Then substitute these values into the quadratic formula and simplify:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \cdot 3 \cdot 7}}{2 \cdot 3} = \frac{3 \pm \sqrt{9 - 84}}{6} = \frac{3 \pm \sqrt{-75}}{6}$$

$$= \frac{3 \pm i\sqrt{75}}{6} = \frac{3 \pm i\sqrt{25 \cdot 3}}{6} = \frac{3 \pm i\sqrt{25} \cdot \sqrt{3}}{6} = \frac{3 \pm i\,5\sqrt{3}}{6}$$

$$x = \frac{3 \pm i\,5\sqrt{3}}{6}$$

Extended Example 2a

Solve: $x^2 + x + 1 = 0$.

EXAMPLE C

Solve $x^2 - 7x + 10 = 0$ using the quadratic formula.

First identify a, b, and c:

$$1x^2 \quad -7x \quad + \quad 10 \quad = \quad 0$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\boxed{a = 1} \; \boxed{b = -7} \; \boxed{c = 10}$$

Substitute these values into the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-7) \pm \sqrt{(-7)^2 - 4 \cdot 1 \cdot 10}}{2 \cdot 1}$$

Simplify:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-7) \pm \sqrt{(-7)^2 - 4 \cdot 1 \cdot 10}}{2 \cdot 1}$$

$$x = \frac{7 \pm \sqrt{49 - 40}}{2} = \frac{7 \pm \sqrt{9}}{2}$$

$$x = \frac{7 \pm 3}{2}$$

continued...

THE QUADRATIC FORMULA

Example C, continued…

We're not quite done. When there's no radical in an answer, we can go a little further. The '+' yields one solution and the '–' yields the other:

$$x = \frac{7+3}{2} = \frac{10}{2} = \boxed{5} \quad \textbf{or} \quad x = \frac{7-3}{2} = \frac{4}{2} = \boxed{2}$$

Note that although the quadratic formula gives the correct solutions, it is easier to solve this particular equation by factoring:

$$\begin{gathered} x^2 - 7x + 10 = 0 \\ (x-5)\cdot(x-2) = 0 \\ \downarrow \qquad\quad \downarrow \\ x = 5 \ \textbf{or} \ x = 2 \end{gathered}$$

A Word to the Wise

If you can see an easy way to factor a quadratic equation, solve it by factoring. Otherwise, bring out the heavy artillery—the quadratic formula.

THE QUADRATIC FORMULA

EXAMPLE D

Solve: $x^2 + 1 = 5x$.

Before we can apply the quadratic formula, we need to get this equation into standard form. Subtract $5x$ from both sides to accomplish this:

$$\begin{array}{rcl} x^2 \quad\quad +1 & = & 5x \\ -5x \quad\quad & & -5x \\ \hline x^2 - 5x + 1 & = & 0 \end{array}$$

Now identify a, b, and c:

$$1x^2 \quad -5x \quad + \quad 1 \quad = \quad 0$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\boxed{a = 1} \; \boxed{b = -5} \; \boxed{c = 1}$$

Substitute these values into the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1}$$

Simplify:

$$x = \frac{5 \pm \sqrt{25 - 4}}{2} \quad \Rightarrow \quad x = \frac{5 \pm \sqrt{21}}{2}$$

THE QUADRATIC FORMULA

EXAMPLE E

Solve: $x(7-2x)=4$.

Before we can apply the quadratic formula, we need to get this equation into standard form. First, we distribute the x to eliminate parentheses. Then get everything on one side of the equal sign to equal 0:

$$x(7-2x) = 4$$

$$7x-2x^2 = 4$$

$$\begin{array}{r} 7x-2x^2 = 4 \\ -4 \quad -4 \\ \hline 7x-2x^2-4 = 0 \end{array}$$

Next, we put the terms of this polynomial in descending order:

$$-2x^2+7x-4=0.$$

We have a little problem here: $a=-2$. In the derivation of the quadratic formula, one step required a to be positive. This is easy to remedy; just multiply both sides of the equation by -1. Use the distributive property and multiply each term by -1:

$$-1\cdot\left(-2x^2+7x-4\right)=-1\cdot 0$$

$$2x^2-7x+4=0$$

continued...

Example E, continued…

Now that our quadratic equation is in standard form, identify a , b , and c :

$$2x^2 \quad -7x \quad + \quad 4 \quad = \quad 0$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\boxed{a = 2} \; \boxed{b = -7} \; \boxed{c = 4}$$

Substitute these values into the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-7) \pm \sqrt{(-7)^2 - 4 \cdot 2 \cdot 4}}{2 \cdot 2}$$

Simplify:

$$x = \frac{7 \pm \sqrt{49 - 32}}{4} \quad \Rightarrow \quad x = \frac{7 \pm \sqrt{17}}{4}$$

Extended Example 3a

Solve: $3x(5-x)=7$.

EXAMPLE F

Find numerical approximations, rounded to the nearest thousandth, for the solutions to $x^2 - 6x + 7 = 0$.

First, identify a, b, and c:

$$1x^2 \quad -6x \quad + \quad 7 \quad = 0$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\boxed{a = 1}\ \boxed{b = -6}\ \boxed{c = 7}$$

Substitute these values into the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 1 \cdot 7}}{2 \cdot 1}$$

This time we only need numerical approximations to the solutions and not the exact answers. We do not have to simplify the expression all the way – only far enough so that we can substitute the numbers into our calculators!

$$x = \frac{6 \pm \sqrt{36 - 28}}{2} \quad \Rightarrow \quad x = \frac{6 \pm \sqrt{8}}{2}$$

continued...

Example F, continued…

First, find the square root of 8. The '+' yields one solution and the '–' yields the other.

$$\sqrt{8} \cong 2.8284271247$$

$$x = \frac{6+\sqrt{8}}{2} \cong \frac{6+2.8284271247}{2} = \frac{8.8284271247}{2} \cong 4.4142135624$$

or

$$x = \frac{6-\sqrt{8}}{2} \cong \frac{6-2.8284271247}{2} = \frac{3.1715728753}{2} \cong 1.5857864376$$

Lastly, we round these answers to the nearest thousandth:

$x \cong 4.414$ **or** $x \cong 1.586$.

Extended Example 4a

Find numerical approximations, rounded to the nearest thousandth, for the solutions to $4x^2 - 5x - 2 = 0$.

THE QUADRATIC FORMULA

The Discriminant

We have seen quadratic equations with real solutions and with complex solutions. The radicand in the quadratic formula determines the type and number of solutions.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This radicand, $b^2 - 4ac$, is called the **discriminant**. Let $D = b^2 - 4ac$. Then...

If $D > 0$, then the quadratic equation has two real solutions.

If $D = 0$, then the quadratic equation has one real solution.

If $D < 0$, then the quadratic equation has two complex solutions.

EXAMPLE G

Use the discriminant to determine the number and type of solutions to the quadratic equation $49x^2 - 70x + 25 = 0$.

As usual, we first identify a, b, and c.

$$49x^2 \quad -70x \quad + \quad 25 \quad = \quad 0$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\boxed{a = 49} \quad \boxed{b = -70} \quad \boxed{c = 25}$$

The discriminant is given by:

$$\begin{aligned} D &= b^2 - 4 \cdot a \cdot c \\ &= (-70)^2 - 4 \cdot 49 \cdot 25 \\ &= 4{,}900 - 4{,}900 \\ D &= 0 \end{aligned}$$

continued...

THE QUADRATIC FORMULA

Example G, continued…

Since the discriminant is zero, we know there is one real solution to this quadratic equation. This makes sense because the quadratic formula then involves "plus or minus 0," which leaves us with $x = \frac{-b}{2a}$. There is only one real solution.

For this example, the solution is:

$$x = \frac{-(-70) \pm \sqrt{0}}{2 \cdot 49} = \frac{70 \pm 0}{98} = \frac{70}{98} = \frac{5}{7}.$$

Note

- The discriminant does not tell us the actual solutions, but only the type and number of them. You must still use the quadratic formula to find the solutions. However, knowing the discriminant allows you to apply the quadratic formula more efficiently.

THE QUADRATIC FORMULA

Extended Example 5a

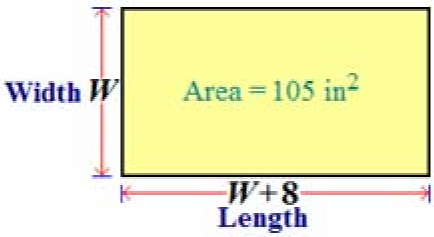
Use the discriminant to determine the number and type of solutions for the quadratic equation $4x^2 + 44x + 121 = 0$.

THE QUADRATIC FORMULA

EXAMPLE H

A rectangle has an area of 105 square inches. The length is 8 inches longer than the width. Find the rectangle's length and the width.

First, a drawing is useful to visualize this. Draw a rectangle and label it with the information given in the problem:



Recall that the area of a rectangle is its length times its width:

$$\text{Area} = \text{Length} \times \text{Width}$$

$$105 = (W + 8) \cdot W$$

Distribute to eliminate parentheses and then subtract 105 from both sides to get this quadratic equation into standard form (but with W in place of x):

$$105 = W^2 + 8W$$

$$-105 \qquad\qquad -105$$

$$0 = W^2 + 8W - 105$$

Identify a, b, and c:

$$1W^2 + 8W \; -105 = 0$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\boxed{a = 1} \; \boxed{b = 8} \; \boxed{c = -105}$$

continued...

THE QUADRATIC FORMULA

Example H, continued…

Substitute these values into the quadratic formula:

$$W=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-8\pm\sqrt{8^2-4\cdot 1\cdot\left(-105\right)}}{2\cdot 1}$$

Simplify:

$$W=\frac{-8\pm\sqrt{64+420}}{2}=\frac{-8\pm\sqrt{484}}{2}=\frac{-8\pm\sqrt{2\cdot 242}}{2}=\frac{-8\pm\sqrt{2\cdot 2\cdot 121}}{2}$$

$$=\frac{-8\pm\sqrt{2^2\cdot 11^2}}{2}=\frac{-8\pm\sqrt{2^2}\cdot\sqrt{11^2}}{2}=\frac{-8\pm 2\cdot 11}{2}=\frac{2\left(-4\pm 11\right)}{2}=\frac{\cancel{2}\left(-4\pm 11\right)}{\cancel{2}}$$

$$W=-4\pm 11$$

We have two possible answers:

$$W=-4+11=\boxed{7}\quad\textbf{or}\quad W=-4-11=\boxed{-15}.$$

Since we are finding the width of a rectangle, a negative solution makes no sense. So, the width is 7 inches while the length is

7 inches + 8 inches = 15 inches.

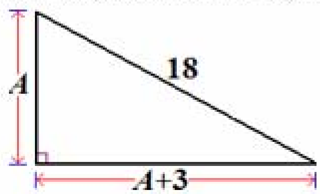
When the solutions don't contain any radicals, the problem could probably be solved more quickly by factoring. Alas, the work has already been done. As in life, experience is often gained right after you needed it!

THE QUADRATIC FORMULA

EXAMPLE I

The hypotenuse of a right triangle measures 18 centimeters in length. One leg is 3 centimeters longer than the other leg. What are the lengths, rounded to the nearest hundredth of a centimeter, of the two legs?

First, make a drawing. Below is a right triangle labeled with the information given in the problem. We have invoked the Pythagorean Theorem to convert the geometric diagram into an algebraic equation:



(Thanks to Pythagoras)

$$A^2 + B^2 = C^2$$

$$A^2 + (A+3)^2 = 18^2$$

Expand the squared binomial and combine like terms:

$$A^2 + (A+3)^2 = 18^2$$

$$A^2 + A^2 + 6A + 9 = 324$$

$$A^2 + A^2 + 6A + 9 = 324$$

$$2A^2 + 6A + 9 = 324$$

Subtract 324 from both sides to get this quadratic equation into standard form:

$$\begin{array}{rcr} 2A^2 + 6A + 9 & = & 324 \\ -324 & & -324 \\ \hline 2A^2 + 6A - 315 & = & 0 \end{array}$$

continued...

THE QUADRATIC FORMULA

Example I, continued…

Identify a, b, and c:

$$2A^2 + 6A \; - 315 \; = \; 0$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\boxed{a = 2} \; \boxed{b = 6} \; \boxed{c = -315}$$

Substitute these values into the quadratic formula:

$$A = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-6 \pm \sqrt{6^2 - 4 \cdot 2 \cdot (-315)}}{2 \cdot 2}$$

Simplify enough to substitute the numbers into a calculator (since we only need a numerical approximation to the exact answer):

$$A = \frac{-6 \pm \sqrt{36 + 2,520}}{4} = \frac{-6 \pm \sqrt{2,556}}{4}$$

We start by calculating $\sqrt{2556}$ and then use that result to evaluate our solutions:

$$\sqrt{2,556} \cong 50.556898639$$

$$A = \frac{-6 + \sqrt{2,556}}{4} \cong \frac{-6 + 50.556898639}{4} = \frac{44.556898639}{4} = 11.1392246598$$

or

$$A = \frac{-6 - \sqrt{2,556}}{4} \cong \frac{-6 - 50.556898639}{4} = \frac{-56.556898639}{4} = -14.139224660$$

continued…

Example I, continued…

$$\sqrt{2,556} \cong 50.556898639$$

$$A = \frac{-6+\sqrt{2,556}}{4} \cong \frac{-6+50.556898639}{4} = \frac{44.556898639}{4} = 11.1392246598$$

or

$$A = \frac{-6-\sqrt{2,556}}{4} \cong \frac{-6-50.556898639}{4} = \frac{-56.556898639}{4} = -14.139224660$$

Rounding these answers to the nearest hundredth, we get:

$$A \cong 11.14 \quad \textbf{or} \quad A \cong -14.14\ .$$

Since we are finding the lengths of sides of a triangle, a negative solution makes no sense!

One leg has length 11.14 cm while the other leg has length

$$11.14 \text{ cm} + 3 \text{ cm} = 14.14 \text{ cm}.$$

THE QUADRATIC FORMULA

Extended Example 6a

A rectangle has an area of 100 square meters. The length is 1 meter longer than the width. Find the dimensions, rounded to the nearest hundredth of a meter, of the rectangle.

END OF LESSON

Calculate the discriminant and use it to determine how many real number solutions there are to: $4x^2 + 5x + 2 = 0$

Solve using the quadratic formula: $4x^2 + 7x + 2 = 0$

Use the Quadratic Formula to solve the equation.

$x(3x - 2) = -7$

Solve using the quadratic formula: $4+\frac{9}{x^2}-\frac{12}{x}=0$

The hypotenuse of a right triangle measures 20 centimeters in length. One leg is 2 centimeters longer than the other leg. What are the lengths of the two legs, rounded to the nearest hundredth of a centimeter?

GRAPHING QUADRATIC EQUATIONS

Introduction

Algebra and geometry are connected in various ways. Using Cartesian coordinates (xy-coordinates), an equation involving the variables x and y can be visualized geometrically. All the points (x, y) that satisfy an equation form its graph. In this lesson, you will learn how to graph quadratic equations, some geometric facts about **parabolas**, and some applications of these topics to real-world problems.

GRAPHING QUADRATIC EQUATIONS

Review

Recall that a number satisfies an equation if a true statement results when the number is substituted into the equation.

In an earlier chapter, you learned that the graph of an equation of the form $y = mx + b$, where m and b are constants, is a straight line. You also learned the geometric significance of those constants; m is the line's slope and b is the line's y-intercept.

In this section, you will see that the graph of a quadratic equation of the form,

$$y = ax^2 + bx + c, \quad \text{where } a,\ b, \text{ and } c \text{ are constants and } a \neq 0,$$

is a **parabola** (an important and useful geometric shape).

Note:

- Any quadratic equation of the form above can be thought of as a function – a **quadratic function**:

$$y = f(x) = ax^2 + bx + c\ .$$

We could have titled this unit "Graphing Quadratic Functions."

GRAPHING QUADRATIC EQUATIONS

EXAMPLE A

Graph: $y = x^2 - 2x - 3$.

As we did with linear equations, we make a table of values for x and y that satisfy the equation.

We will need more points to graph quadratic equations than we did to graph linear equations since the graph of a quadratic equation is a curve, not a straight line.

To find points on the graph, let x take on a range of values:

x	y
-2	
-1	
0	
1	
2	
3	
4	

For each x value, we seek the unique y value that will make the equation true.

To do this, we must substitute each x value into the equation.

continued...

GRAPHING QUADRATIC EQUATIONS

Example A, continued…

Thinking of the equation as a function, we substitute a few inputs into the function to find the corresponding outputs:

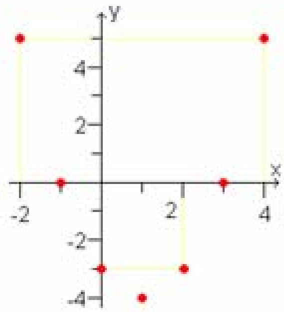
$y = x^2 - 2x - 3$	x	y
$y = (-2)^2 - 2(-2) - 3$ $= 4 + 4 - 3 = 5$	-2	5
$y = (-1)^2 - 2(-1) - 3$ $= 1 + 2 - 3 = 0$	-1	0
$y = 0^2 - 2 \cdot 0 - 3$ $= 0 - 0 - 3 = -3$	0	-3
$y = 1^2 - 2 \cdot 1 - 3$ $= 1 - 2 - 3 = -4$	1	-4
$y = 2^2 - 2 \cdot 2 - 3$ $= 4 - 4 - 3 = -3$	2	-3
$y = 3^2 - 2 \cdot 3 - 3$ $= 9 - 6 - 3 = 0$	3	0
$y = 4^2 - 2 \cdot 4 - 3$ $= 16 - 8 - 3 = 5$	4	5

continued…

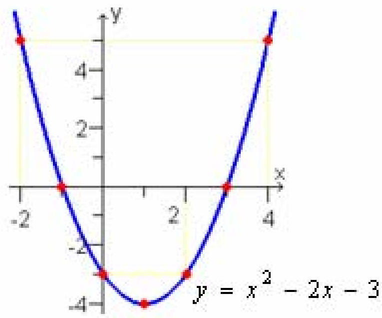
GRAPHING QUADRATIC EQUATIONS

Example A, continued…

Plotting these points, we get:



Smoothly connecting these points, we see a parabola emerge:



The curve above is an example of a **parabola.** All quadratic equations of the form $y = ax^2 + bx + c$, where $a \neq 0$, have graphs that are parabolas. If you turn the parabola above upside down, it has the shape of the path of a projectile flying through the air. When you throw a baseball into the air, it follows a parabolic path (until it hits something!). Keep this in mind when you "connect the dots" to graph parabolas.

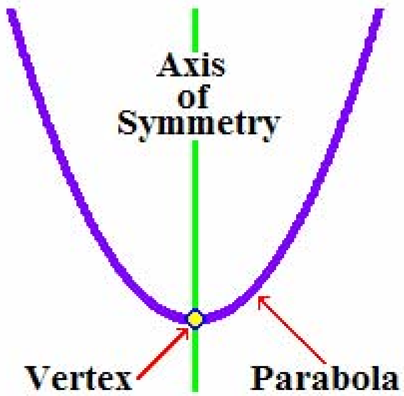
You may find it helpful to view this animation of Example A:

GRAPHING QUADRATIC EQUATIONS

Although we plotted quite a few points in Example A, it turns out that three points (not all on the same line) uniquely determine a parabola. In practice, one can plot a reasonably accurate parabola with only three points—if they are the right three points!

Notice in the diagram below that parabolas have **bilateral symmetry**: a parabola has a left side and a right side that are mirror images of each other. There is a vertical line through the center of the parabola called the **axis of symmetry** of the parabola. This vertical line passes through the lowest (or the highest) point of the parabola. This special point is called the **vertex** of the parabola, from the Latin verb "vertare" meaning to turn.

In the graph of Example A, the vertex is $(1,-4)$, the point where the parabola turned from going down to going up (moving from left to right).



GRAPHING QUADRATIC EQUATIONS

The vertex is the single most important point on a parabola. The x-coordinate of the vertex of the parabola given by the graph of $y = ax^2 + bx + c$ is $\boxed{x = \frac{-b}{2a}}$.

We can use this formula to find the x-coordinate of the vertex of any parabola. We can then find the corresponding y-coordinate by plugging the x-coordinate into the parabola's equation.

Notice that the x-coordinate of the vertex is part of the quadratic formula. If you ignore the radical part of the formula (grayed out in the image below), you can see the x-intercept:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Knowing the quadratic formula thus makes it easy to recall the x-coordinate of the vertex.

The vertex of the parabola defined by a quadratic equation $y = ax^2 + bx + c$ is either the highest or the lowest point on the parabola. Thus, a quadratic function $f(x) = ax^2 + bx + c$ attains its **maximum** or **minimum** at the vertex.

Notice that the x-coordinate of the vertex, $x = \frac{-b}{2a}$, is also the equation of a vertical line. This isn't just any vertical line. Recall that the axis of symmetry of the parabola is a vertical line passing through the parabola's vertex. It follows that $x = \frac{-b}{2a}$ refers not only to the x-coordinate of the vertex but also to the equation of the parabola's axis of symmetry.

GRAPHING QUADRATIC EQUATIONS

EXAMPLE B

Find the vertex and axis of symmetry of the parabola $y = x^2 - 2x - 3$.

First, identify a, b, and c:

$$y = 1x^2 \quad -2x \quad -3$$

$$\boxed{a = 1} \quad \boxed{b = -2} \quad \boxed{c = -3}$$

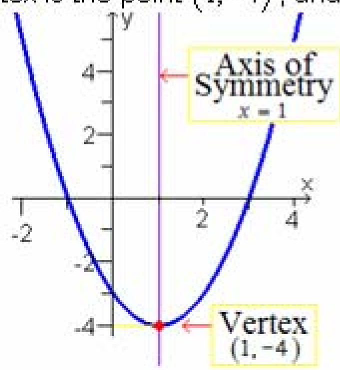
The x-coordinate of the vertex, as well as the equation for the axis of symmetry, is given by:

$$x = \frac{-b}{2a} = \frac{-(-2)}{2 \cdot 1} = \frac{2}{2} = 1$$

Now that we have the x-coordinate of the vertex, we can find the corresponding y-coordinate by substituting the x-coordinate into the equation:

$$\begin{aligned} y &= x^2 - 2x - 3 \\ &= 1^2 - 2 \cdot 1 - 3 \\ &= 1 - 2 - 3 = -4 \end{aligned}$$

The vertex is the point $(1, -4)$, and the axis of symmetry is the vertical line $x = 1$:



$$y = x^2 - 2x - 3$$

GRAPHING QUADRATIC EQUATIONS

Extended Example 1a

Find the vertex and axis of symmetry of the parabola $y = 3x^2 + 12x + 5$.

GRAPHING QUADRATIC EQUATIONS

Later in this section we will look at how each of the constants, a , b , and c of the equation $y = ax^2 + bx + c$ affect the graph of the parabola. But for now, it's very helpful to know one important fact - that the sign of a determines whether the parabola opens upwards or downwards.

If $a > 0$, then the parabola opens upwards. If $a < 0$, the parabola opens downwards. If $a = 0$, then the graph isn't a parabola (it's a line). So to see if the parabola opens upwards or downwards, just ask a :

$$y = ax^2 + bx + c$$

$a < 0$ $\quad$ $a = 0$ $\quad$ $a > 0$

Now that you know how to find the vertex of a parabola, it will be much easier to graph parabolas.

GRAPHING QUADRATIC EQUATIONS

EXAMPLE C

Graph the parabola defined by the equation $y = 3x^2 - 12x - 4$.

First, identify a, b, and c:

$$y = 3x^2 \quad -12x \quad -4$$
$$\downarrow \qquad \downarrow \qquad \downarrow$$
$$\boxed{a = 3} \ \boxed{b = -12} \ \boxed{c = -4}$$

Next, we will find the vertex. The x-coordinate of the vertex (which is also the equation of the axis of symmetry) is:

$$x = \frac{-b}{2a} = \frac{-(-12)}{2 \cdot 3} = \frac{12}{6} = 2.$$

To find the y-coordinate of the vertex, substitute this value of x into the equation:

$$\begin{aligned} y &= 3x^2 - 12x - 4 \\ &= 3 \cdot 2^2 - 12 \cdot 2 - 4 \\ &= 3 \cdot 4 - 24 - 4 \\ &= 12 - 28 \\ y &= -16 \end{aligned}$$

The vertex is the point $(2, -16)$. Let's find some more points on this parabola in order to graph it.

continued...

GRAPHING QUADRATIC EQUATIONS

Example C, continued…

The y-intercept is the easiest one to find on any parabola –it is the only point with an x-coordinate of zero:

$$\begin{aligned} y &= 3x^2 - 12x - 4 \\ &= 3\cdot 0^2 - 12\cdot 0 - 4 \\ &= 0 - 0 - 4 \\ y &= -4 \end{aligned}$$

Thus, the y-intercept of this parabola is the point $(0,-4)$.

For one more point, let's try $x = 3$. Then:

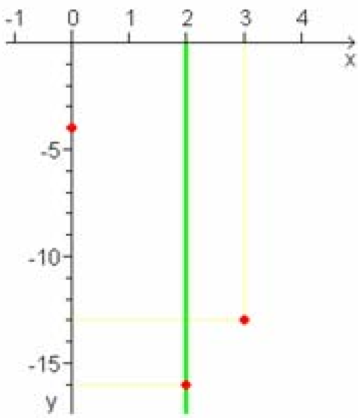
$$\begin{aligned} y &= 3x^2 - 12x - 4 \\ &= 3\cdot 3^2 - 12\cdot 3 - 4 \\ &= 3\cdot 9 - 36 - 4 \\ &= 27 - 40 \\ y &= -13 \end{aligned}$$

Thus, the point $(3,-13)$ is on the parabola.

continued…

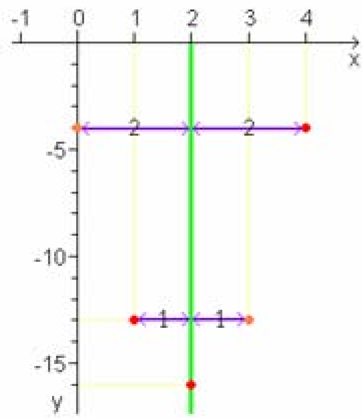
Example C, continued…

Now we will begin graphing this parabola. Start by plotting the points on the parabola, $(2,-16)$, $(0,-4)$, and $(3,-13)$. Also, we will draw the axis of symmetry, the vertical line $x = 2$:



For every point on the parabola that is on the one side of the line of symmetry, there is a point on the opposite side, an equal distance away.

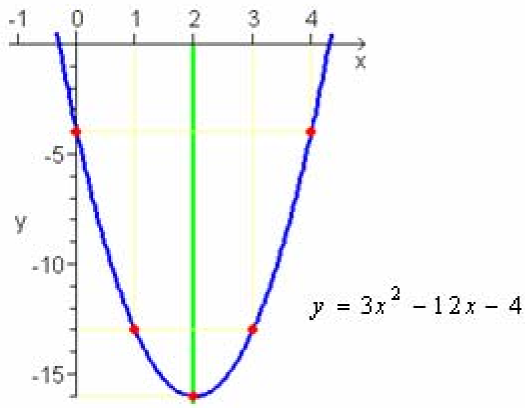
This gives us a total of five points on the parabola, including the vertex.



continued…

Example C, continued…

We can draw in the parabola that passes through the plotted points. Remember that the shape is the path of a thrown ball (as seen standing on your head, since this parabola opens upward!):



Note:

- The axis of symmetry is <u>not</u> part of the parabola's graph. It's drawn in to make it easier to draw the parabola. As you have seen, the symmetry gives us two points for the price of one!

GRAPHING QUADRATIC EQUATIONS

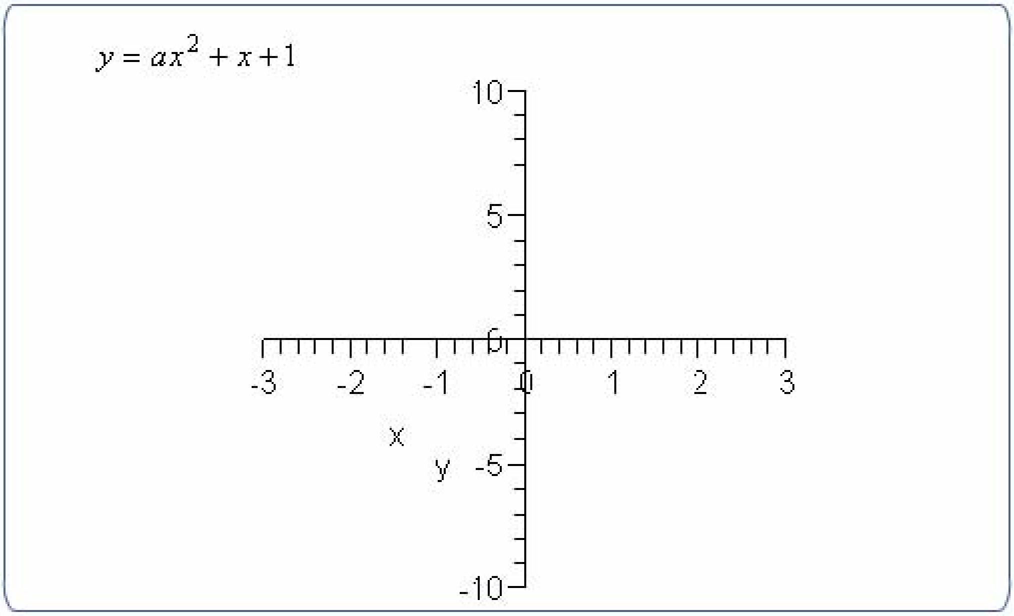
Extended Example 2a

Graph the parabola defined by the equation $y = x^2 + 2x - 7$.

GRAPHING QUADRATIC EQUATIONS

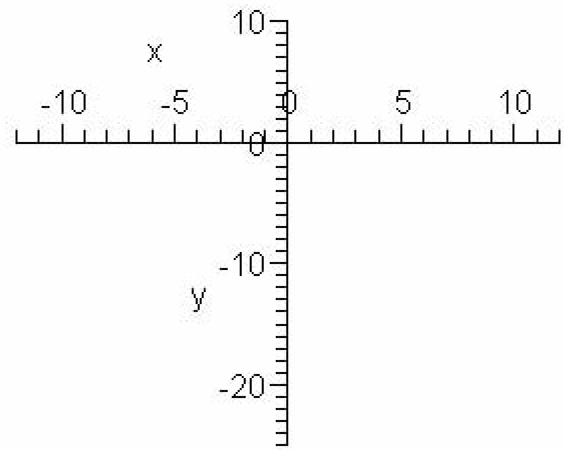
Each of the constants, a , b , and c , in the equation $y = ax^2 + bx + c$ affect the graph of the parabola. These are like knobs that can be adjusted with each possible setting resulting in a different parabola. Now, we will take a look at each of these **parameters**.

We've seen how the sign of a determines whether the parabola opens upwards or downwards. This coefficient also determines how skinny the parabola is. If a is small in magnitude, the parabola is fat, whereas if a is large in magnitude, the parabola is skinny.



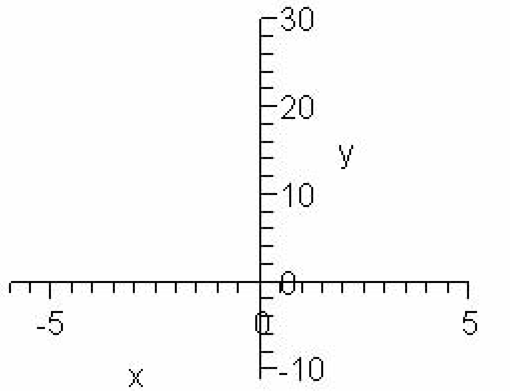
GRAPHING QUADRATIC EQUATIONS

$$y = x^2 + bx + 1$$



GRAPHING QUADRATIC EQUATIONS

$y = x^2 + x + c$



Finding a Parabola's x-Intercepts

In graphing parabolas, you may have noticed that it would be helpful to know the parabola's x-intercepts. Where exactly does a parabola intersect the x-axis? For lines, we found the x-intercept by setting $y = 0$, and then solving for x. Finding a parabola's x-intercepts is done in the same way.

EXAMPLE D

Find the x-intercepts of the parabola defined by $y = 3x^2 - 12x - 4$. Find the exact answers and decimal approximations accurate to the nearest tenth.

To find the x-intercept, we start by setting $y = 0$: $0 = 3x^2 - 12x - 4$.

Now we solve for x using the quadratic formula. Since $a = 3$, $b = -12$, and $c = -4$, we have:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-12) \pm \sqrt{(-12)^2 - 4 \cdot 3 \cdot (-4)}}{2 \cdot 3} = \frac{12 \pm \sqrt{144 + 48}}{2 \cdot 3}$$

$$= \frac{12 \pm \sqrt{192}}{2 \cdot 3} = \frac{12 \pm \sqrt{2^6 \cdot 3}}{2 \cdot 3} = \frac{12 \pm 2^3\sqrt{3}}{2 \cdot 3} = \frac{12 \pm 8\sqrt{3}}{2 \cdot 3}$$

$$= \frac{4 \cdot \left(3 \pm 2\sqrt{3}\right)}{2 \cdot 3} = \frac{\cancel{2} \cdot 2 \cdot \left(3 \pm 2\sqrt{3}\right)}{\cancel{2} \cdot 3}$$

$$x = \frac{2\left(3 \pm 2\sqrt{3}\right)}{3}$$

continued...

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GRAPHING QUADRATIC EQUATIONS

Example D, continued…

The x-intercepts are the points $\left(\frac{2\left(3+2\sqrt{3}\right)}{3},0\right)$ and $\left(\frac{2\left(3-2\sqrt{3}\right)}{3},0\right)$.

Rounding to the nearest tenth, we get:

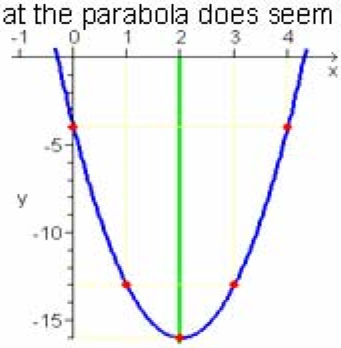
$$\frac{2\left(3+2\sqrt{3}\right)}{3} \cong \frac{2\left(3+2\left(1.73205080757\right)\right)}{3} = \frac{2\left(3+3.46410161514\right)}{3}$$

$$\cong \frac{2\left(6.46410161514\right)}{3} = \frac{12.928203230}{3} = 4.30940107675$$

$$\frac{2\left(3-2\sqrt{3}\right)}{3} \cong \frac{2\left(3-2\left(1.73205080757\right)\right)}{3} = \frac{2\left(3-3.46410161514\right)}{3}$$

$$\cong \frac{2\left(-0.464101615138\right)}{3} = \frac{-0.92820323028}{3} = -0.30940107675$$

The x-intercepts are approximately the points: $\left(4.3,0\right)$ and $\left(-0.3,0\right)$. The equation in this example is from Example C. Re-examining Example C's graph, we see that the parabola does seem to cross the x-axis at precisely these points:



$y = 3x^2 - 12x - 4$

GRAPHING QUADRATIC EQUATIONS

Note:

- When you solve for x and don't get real solutions, you have a parabola that doesn't cross the x-axis, so there are no x-intercepts.

Extended Example 3a

Find the x-intercepts of the parabola defined by $y = x^2 + 8x - 3$. Find the exact answers and decimal approximations accurate to the nearest tenth.

EXAMPLE E

The equation $h = -32.2t^2 + 1,000t$ describes the height h, in feet, of a bullet fired from a gun up into the air t seconds after the bullet was fired. How long will it take the bullet to get to its highest point, and how high will it go?

The equation defines a parabola (think of t as x and h as y). Recall that the vertex of a parabola is its highest (or lowest) point. In this case, the parabola opens downwards (due to the negative a coefficient). Thus, the vertex is the highest point. Let's find this vertex! This time, $a = -32.2,\ b = 1,000,$ and $c = 0$, and the t-coordinate of the vertex is given by:

$$t = \frac{-b}{2a} = \frac{-1000}{2\cdot(-32.2)} = \frac{-1000}{-64.4} \cong 15.5279503106 .$$

A little over 15 and one-half seconds pass before the bullet attains its maximum height. The maximum height can be obtained by substituting this value into the height equation:

$$\begin{aligned} h &= -32.2(15.5279503106)^2 + 1,000(15.5279503106) \\ &= -32.2(241.11724084) + 15,527.9503106 \\ &= -7,763.9751553 + 15,527.9503106 \\ h &= 7,763.9751553 \end{aligned}$$

The bullet will attain a height of about 7,764.0 feet.

EXAMPLE F

The equation $P = -16n^2 + 64n$ describes the profit P in thousands of dollars, when n thousand units are manufactured (per day). How many units should be manufactured to maximize the profit, and what is the maximum daily profit?

The equation defines a parabola (think of n as x and P as y). Recall that the vertex of a parabola is its highest (or lowest) point. In this case, the parabola opens downwards (due to the negative a coefficient). The vertex is the highest point. Let's find this vertex! This time, $a = -16$, $b = 64$, and $c = 0$, so the n-coordinate of the vertex is given by:

$$n = \frac{-b}{2a} = \frac{-64}{2 \cdot (-16)} = \frac{-64}{-32} = 2.$$

Recall that n is the number of thousands of units and, therefore, manufacturing 2,000 units will maximize the profit. The maximum profit can be obtained by substituting $n = 2$ into the profit equation:

$$\begin{aligned} P &= -16n^2 + 64n \\ &= -16 \cdot 2^2 + 64 \cdot 2 \\ &= -64 + 128 \\ P &= 64 \end{aligned}$$

Since P is measured in thousands of dollars, the maximum daily profit is $64,000.

GRAPHING QUADRATIC EQUATIONS

EXAMPLE G

The area enclosed by a corral is given by $A = -2w^2 + 60w$, where w is the width of the corral in meters and A is measured in square meters. What width will yield the maximum area, and what is that maximum area?

The equation defines a parabola (think of w as x and A as y). Recall that the vertex of a parabola is its highest (or lowest) point. In this case, the parabola opens downwards (due to the negative a coefficient). The vertex is the highest point. Let's find this vertex! This time, $a = -2,\ b = 60,$ and $c = 0$. The w-coordinate of the vertex is given by:

$$w = \frac{-b}{2a} = \frac{-60}{2 \cdot (-2)} = \frac{-60}{-4} = 15.$$

Recall that w is measured in meters. Thus, a width of 15 meters will maximize the area. The maximum area can be obtained by substituting $w = 15$ into the area equation:

$$\begin{aligned} A &= -2w^2 + 60w \\ &= -2 \cdot 15^2 + 60 \cdot 15 \\ &= -2 \cdot 225 + 900 \\ &= -450 + 900 \\ A &= 450 \end{aligned}$$

Since A is measured in square meters, the maximum area is 450 square meters.

GRAPHING QUADRATIC EQUATIONS

Extended Example 4a

The equation $h = -32.2t^2 + 1,200t$ describes the height h, in feet, of a bullet fired from a gun up into the air t seconds after the bullet was fired. How long will it take the bullet to get to its highest point, and how high will it go? Round the time to the nearest tenth of a second and round the height to the nearest foot.

Summary

1. The graph of a quadratic equation $y = ax^2 + bx + c$, when $a \neq 0$, is a parabola.
2. A parabola is symmetrical. A vertical axis of symmetry runs through its vertex.
3. A parabola opens upward if $a > 0$, and opens downward if $a < 0$.
4. The larger a is in absolute value, the narrower the parabola.
5. The c parameter moves the graph of a parabola up or down.
6. The lowest or highest point of a parabola is called its vertex. The x-coordinate of the vertex is $\frac{-b}{2a}$.

END OF LESSON

Find the equation of the axis of symmetry of the parabola:

$y = -3x^2 + 11x + 3$

Find the coordinate of the vertex of the parabola:

$y = -3x^2 + 11x + 3$

Graph: $y = 10 + 3x - x^2$

Graph.

$y = 3 + 2x - x^2$

The area enclosed by a corral is given by $A = -4w^2 + 60w$, where *w* is the width of the corral in meters and A is measured in square meters. What width will yield the maximum area, and what is that maximum area?