MULTIPLYING AND DIVIDING RADICAL EXPRESSIONS

Introduction

After completing this section of the course, you will know how to simplify, multiply, and divide radical expressions involving square roots, cube roots, and n^{th} roots. You will also gain an understanding of the algebraic rules for radicals.

MULTIPLYING AND DIVIDING RADICAL EXPRESSIONS

Review

Recall that $\sqrt{4x}$ is a simple example of a **radical expression**. The symbol $\sqrt{\ }$ is called a **radical**, and the number or expression under the radical is called the **radicand**.

Suppose that A is a non-negative real number. Then we can write

$$\sqrt{A} = B \text{ whenever it is true that } A = B^2.$$

So it is correct to write $\sqrt{4} = 2$, since it is true that $4 = 2^2$.

Notice that when A is negative, a new situation is presented: $\sqrt{A} = B$ means that $A = B^2$. However, A cannot be negative if it equals B^2, since B^2 cannot be negative (B^2 is positive for any real number B). The square root of a negative number is not a real number.

The term **square root** comes from the observation that a square with area A has side lengths all equal to $\sqrt{A}$.

If A represents a nonnegative quantity, then $\sqrt{A^2} = A$.
It is important to realize that both sides of this equation are positive. But what happens if A represents a negative quantity? Or what happens if A represents both positive and negative values? Then we can say that

$$\sqrt{A^2} = |A|.$$

The absolute value on the right side of the equation makes the equation true no matter what real number A may represent, since absolute value is positive.

MULTIPLYING AND DIVIDING RADICAL EXPRESSIONS

The following properties are very useful in simplifying, multiplying, and dividing square roots. The Roman numerals will be used later to refer to each rule. Note that these rules assume that the variables represent non-negative quantities.

Algebraic Properties of the Square Root If $A \geq 0$ and $B \geq 0$, then:		
I	$\sqrt{A}\sqrt{B} = \sqrt{AB}$	"The product of radicals is the radical of the product."
II	$\dfrac{\sqrt{A}}{\sqrt{B}} = \sqrt{\dfrac{A}{B}}, \quad B \neq 0$	"The quotient of radicals is the radical of the quotient."
III	$\sqrt{A^2} = \sqrt{A}\sqrt{A} = \left(\sqrt{A}\right)^2 = A$	"Squaring a square root eliminates the radical."

First we will focus on using these rules to simplify square roots. Refer to these rules as you read through the following examples. Remember that the formulas above may be read from left to right, or from right to left.

Note:

- In this section of the course, all variables represent non-negative quantities, unless otherwise stated.

MULTIPLYING AND DIVIDING RADICAL EXPRESSIONS

Simplifying Square Roots

The three algebraic properties of the square root are the keys to simplifying square roots. The main idea behind the simplification of square roots is part of Rule III,

$$\sqrt{A^2} = A,$$

for non-negative values of A. Taking the square root of a squared quantity eliminates the square. In other words, the radical symbol cancels out the 2 exponent. Finding the square root of a perfect square is an easy simplification.

Using Rule I, $\sqrt{A}\sqrt{B} = \sqrt{AB}$, we can also simplify the square root of a number that has a perfect square factor, as shown in the next example.

EXAMPLE A

Simplify: $\sqrt{75}$.

The radicand can be written as the perfect square 25, multiplied by 3. Applying Rule I allows us to then apply Rule III, thus simplifying the radical:

$$\begin{aligned}\sqrt{75} &= \sqrt{25 \cdot 3} \\ &\overset{\text{I}}{=} \sqrt{25} \cdot \sqrt{3} \\ &= \sqrt{5^2} \cdot \sqrt{3} \\ &\overset{\text{III}}{=} 5\sqrt{3}\end{aligned}$$

Extended Example 1a

Simplify: $\sqrt{1700}$.

MULTIPLYING AND DIVIDING RADICAL EXPRESSIONS

Often it is necessary to find the prime factorization of the radicand first, and then apply the square root properties to simplify it.

EXAMPLE B

Simplify: $\sqrt{720}$.

First, we calculate the prime factorization of 720:

$$\begin{aligned} 720 &= 72 \cdot 10 \\ &= 8 \cdot 9 \cdot 2 \cdot 5 \\ &= 8 \cdot 9 \cdot 2 \cdot 5 \\ &= 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 2 \cdot 5 \\ &= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \\ &= 2^4 \cdot 3^2 \cdot 5 \end{aligned}$$

Then, using Rule I, $\sqrt{A}\sqrt{B} = \sqrt{AB}$, we get:

$$\sqrt{720} = \sqrt{2^4 \cdot 3^2 \cdot 5} \overset{\mathbf{I}}{=} \sqrt{2^4} \cdot \sqrt{3^2} \cdot \sqrt{5}$$

Notice that $2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 4 \cdot 4 = 4^2$.

Using Rule III, $\sqrt{A^2} = A$:

$$= \sqrt{4^2} \cdot \sqrt{3^2} \cdot \sqrt{5} \overset{\mathbf{III}}{=} 4 \cdot 3 \cdot \sqrt{5} = 12\sqrt{5} .$$

MULTIPLYING AND DIVIDING RADICAL EXPRESSIONS

Study the square roots in Example B on the previous screen. Notice that

$$\sqrt{2^4} \cdot \sqrt{3^2} = 2^2 \cdot 3^1 .$$

Taking the square root in each case cut each exponent in half. If you remember that the radical symbol , $\sqrt{}$,resembles the long division symbol , $\overline{)}$, you can think of a square root as the division of the exponent by 2. Since square roots divide exponents by 2, they are sometimes written with a 2: $\sqrt{A} = \sqrt[2]{A}$. The 2 is usually not written, but it's helpful to think of it as being there.

Using Rule I, $\sqrt{A}\sqrt{B} = \sqrt{AB}$, you can write a product under one radical, or break it up into a product of radicals:

$$\sqrt{2^{12} \cdot 3^{16} \cdot 5^{22}} = \sqrt{2^{12}} \cdot \sqrt{3^{16}} \cdot \sqrt{5^{22}} = 2^6 \cdot 3^8 \cdot 5^{11} .$$

So we could have written: $\sqrt[2]{2^{12} \cdot 3^{16} \cdot 5^{22}} = 2^6 \cdot 3^8 \cdot 5^{11}$.

This works equally well with non-negative variables:

$$\sqrt[2]{a^{12} \cdot b^{16} \cdot c^{22}} = a^6 \cdot b^8 \cdot c^{11} .$$

Simplifying a square root is as simple as dividing an exponent by 2! This is a consequence of Rule III, $\sqrt{A^2} = A$, along with the exponent rule $\left(A^m\right)^n = A^{m \cdot n}$ *:

$$\sqrt[2]{a^{12} \cdot b^{16} \cdot c^{22}} \overset{*}{=} \sqrt[2]{\left(a^6\right)^2 \cdot \left(b^8\right)^2 b \cdot \left(c^{11}\right)^2} \overset{\text{III}}{=} a^6 \cdot b^8 \cdot c^{11} .$$

MULTIPLYING AND DIVIDING RADICAL EXPRESSIONS

EXAMPLE C

Simplify: $\sqrt{56a^6b^{14}}$.

First, we calculate the prime factorization of 56:

$$56 = 8 \cdot 7 = 2 \cdot 2 \cdot 2 \cdot 7 = 2^3 \cdot 7$$

Using Rule I, $\sqrt{A}\sqrt{B} = \sqrt{AB}$, and then Rule III, $\sqrt{A^2} = A$, we get:

$$\sqrt{56a^6b^{14}} = \sqrt{2^3 \cdot 7 \cdot a^6 \cdot b^{14}} \overset{\mathbf{I}}{=} \sqrt{2^3} \cdot \sqrt{7} \cdot \sqrt{a^6} \cdot \sqrt{b^{14}}$$

$$\overset{\mathbf{III}}{=} \sqrt{2^3} \cdot \sqrt{7} \cdot a^3 \cdot b^7$$

We can use Rule I one more time, if we first write $\sqrt{2^3} = \sqrt{2^2 \cdot 2}$:

$$= \sqrt{2^3} \cdot \sqrt{7} \cdot a^3 \cdot b^7 = \sqrt{2^2 \cdot 2} \cdot \sqrt{7} \cdot a^3 \cdot b^7$$

$$\overset{\mathbf{I}}{=} \sqrt{2^2} \cdot \sqrt{2} \cdot \sqrt{7} \cdot a^3 \cdot b^7$$

$$= \sqrt{2^2} \cdot \sqrt{2} \cdot \sqrt{7} \cdot a^3 \cdot b^7$$

$$\overset{\mathbf{III}}{=} 2 \cdot \sqrt{2} \cdot \sqrt{7} \cdot a^3 \cdot b^7$$

Lastly, combine the two radicals under one radical using Rule I and put the resulting radical to the right of the other factors (the preferred order for products involving a radical):

$$= 2 \cdot a^3 \cdot b^7 \cdot \sqrt{2} \cdot \sqrt{7} \overset{\mathbf{I}}{=} 2a^3b^7\sqrt{14} \, .$$

MULTIPLYING AND DIVIDING RADICAL EXPRESSIONS

In Example C, you may have wondered what to do about the $\sqrt{2^3}$, since 3 is not divisible by 2. We rewrote it as

$$\sqrt{2^3} = \sqrt{2^2 \cdot 2} = \sqrt{2^2} \cdot \sqrt{2} = 2\sqrt{2}.$$

Look carefully at the process. When we divide 3 by 2 we think, "Two goes into three once, with a remainder of one." Look again at what we did. We can write in the exponents of 1, since for all real numbers, $a = a^1$:

$$\sqrt[2]{2^3} = 2^1\sqrt{2^1}$$

"Two goes into three once, with a remainder of one"

From now on, we will simplify radicals in this simple way. Notice how the remainder <u>remains</u> under the radical.

You will need to use a couple of exponent rules, so you may want to review these exponent properties:

$$a^0 = 1$$

$$a^1 = a$$

Also, we will use Rule I without always stating it. For example:

$$\sqrt{a \cdot b \cdot c \cdot c} = \sqrt{a} \cdot \sqrt{b} \cdot \sqrt{c} \cdot \sqrt{c}$$

You can take the square root of a product by taking the square root of each factor. It's often easier to just have one radical over everything instead of having to write out each separate radical.

MULTIPLYING AND DIVIDING RADICAL EXPRESSIONS

Extended Example 2a

Simplify: $\sqrt{288u^{29}v^{33}}$.

MULTIPLYING AND DIVIDING RADICAL EXPRESSIONS

So far we have not used the second property of square roots, "The radical of a quotient is the quotient of the radicals." This property is examined below.

EXAMPLE D

Simplify: $\sqrt{\dfrac{18}{25}}$.

First, use Rule II, $\dfrac{\sqrt{A}}{\sqrt{B}} = \sqrt{\dfrac{A}{B}}$, $B \neq 0$, to transform the radical of a quotient to a quotient of radicals:

$$\sqrt{\frac{18}{25}} \overset{\text{II}}{=} \frac{\sqrt{18}}{\sqrt{25}}$$

Factor the radicands, and then use Rule I to write a radical of a product as the product of radicals:

$$= \frac{\sqrt{2 \cdot 3^2}}{\sqrt{5^2}} \overset{\text{I}}{=} \frac{\sqrt{2} \cdot \sqrt{3^2}}{\sqrt{5^2}}$$

Finally, use Rule III and then reorder the terms with the radical on the right:

$$= \frac{\sqrt{2} \cdot \sqrt{3^2}}{\sqrt{5^2}} \overset{\text{III}}{=} \frac{\sqrt{2} \cdot 3}{5} = \frac{3\sqrt{2}}{5}$$

Extended Example 3a

Simplify: $\frac{\sqrt{24}}{\sqrt{600}}$.

Extended Example 4a

Simplify: $\sqrt{\dfrac{x^{13}z^{27}}{y^{16}}}$.

MULTIPLYING AND DIVIDING RADICAL EXPRESSIONS

EXAMPLE E

Simplify: $\sqrt{30w^5v^7}\sqrt{24w^{11}v^3}$.

We can work out each radical separately, but it's easier to combine them under one radical. Do not multiply all of the numbers, though, since we will need those numbers later in the prime factorizations. So, we first factor the radicands' coefficients, and then combine the radicands under one radical (Rule I):

$$\begin{aligned}\sqrt{30w^5v^7}\sqrt{24w^{11}v^3} &= \sqrt{2\cdot 3\cdot 5\cdot w^5v^7}\sqrt{2^3\cdot 3\cdot w^{11}v^3}\\ &= \sqrt{2^4\cdot 3^2\cdot 5\cdot w^5\cdot w^{11}\cdot v^7\cdot v^3}\\ &= \sqrt{2^4\cdot 3^2\cdot 5\cdot w^{5+11}\cdot v^{7+3}}\\ &= \sqrt{2^4\cdot 3^2\cdot 5^1\cdot w^{16}\cdot v^{10}}\end{aligned}$$

Now divide each exponent by 2. In each case, the remainder remains under the radical:

$$\begin{aligned}&= 2^2\cdot 3^1\cdot w^8\cdot v^5\cdot \sqrt{5^1}\\ &= 12w^8v^5\sqrt{5}\end{aligned}$$

14 of 27

EXAMPLE F

Simplify: $\sqrt{30w^5v^7} \div \sqrt{24w^{11}v^3}$.

Start by writing this problem as a numerator over a denominator, and factor the coefficients. Then use Rule II (so we can cancel the common factors):

$$\frac{\sqrt{30w^5v^7}}{\sqrt{24w^{11}v^3}} \overset{\text{II}}{=} \sqrt{\frac{2\cdot 3\cdot 5\cdot w^5v^7}{2^3\cdot 3\cdot w^{11}v^3}}$$

Now we can cancel. It is simplest if we use the exponent rule, $\frac{a^m}{a^n} = a^{m-n}$.

$$\sqrt{\frac{2^1\cdot 3\cdot 5\cdot w^5\cdot v^7}{2^3\cdot 3\cdot w^{11}\cdot v^3}} = \sqrt{\frac{2^{1-3}\cdot \cancel{3}\cdot 5\cdot w^{5-11}\cdot v^{7-3}}{\cancel{3}}}$$

$$= \sqrt{\frac{2^{-2}\cdot 5\cdot w^{-6}\cdot v^4}{1}}$$

Eliminate the negative exponents using the "elevator rule":

$$= \sqrt{\frac{5\cdot v^4}{2^2\cdot w^6}}$$

Finally, use Rule II and then divide each exponent by 2:

$$= \frac{\sqrt{5^1\cdot v^4}}{\sqrt{2^2\cdot w^6}} = \frac{\sqrt[2]{5^1\cdot v^4}}{\sqrt[2]{2^2\cdot w^6}}$$

$$= \frac{v^2\cdot \sqrt{5^1}}{2^1\cdot w^3} = \frac{v^2\sqrt{5}}{2w^3}$$

Extended Example 5a

Simplify: $\sqrt{28x^7y^3}\sqrt{24x^9y^5}$.

Rationalizing Denominators (Part I)

Sometimes a square root remains in the denominator after simplifying an expression. An expression is considered more simplified if there are no radicals in the denominator. A final step is sometimes needed to remove any unwanted radicals from the denominator. That step can sometimes be accomplished using Rule III; in particular, that part of Rule III that says that $\sqrt{A} \cdot \sqrt{A} = A$. Our next set of examples illustrates the process of **rationalizing the denominator**.

EXAMPLE G

Simplify: $\sqrt{6x^3} \div \sqrt{42x}$.

$$\frac{\sqrt{6x^3}}{\sqrt{42x}} \overset{\text{II}}{=} \sqrt{\frac{2 \cdot 3 \cdot x^3}{2 \cdot 3 \cdot 7 \cdot x^1}}$$

$$= \sqrt{\frac{\cancel{2 \cdot 3} \cdot x^{3-1}}{\cancel{2 \cdot 3} \cdot 7}}$$

$$= \sqrt{\frac{x^2}{7}}$$

$$\overset{\text{II}}{=} \frac{\sqrt{x^2}}{\sqrt{7}}$$

$$\overset{\text{III}}{=} \frac{x}{\sqrt{7}}$$

continued...

Example G, continued...

To eliminate $\sqrt{7}$ from the denominator, simply multiply the numerator and denominator by radical seven:

$$= \frac{x}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}}$$

Since the $\sqrt{7} \cdot \sqrt{7} = 7$ (Rule III), the radical is then removed from the denominator:

$$= \frac{x \cdot \sqrt{7}}{\sqrt{7} \cdot \sqrt{7}} \overset{\text{III}}{=} \frac{x\sqrt{7}}{7}$$

Extended Example 6a

Rationalize the denominator: $\frac{x\sqrt{3}}{\sqrt{5x}}$.

Cube Roots

The **cube root** of A written $\sqrt[3]{A}$, is that particular number whose cube is A. Suppose that A is any real number (for the moment we will also include negative numbers). Then we write

$$\sqrt[3]{A} = B \text{ whenever it is true that } A = B^3.$$

For example,

$$\sqrt[3]{8} = 2 \text{, since } 8 = 2^3.$$

Note:

- The reason the term "cube root" is used is that if a cube has volume V, then its edge length is $\sqrt[3]{V}$. For example, a cube with volume 8 cubic centimeters has edge length 2 centimeters.

Although the square roots of negative numbers are not real numbers, the cube root of every real number is a real number. So, $\sqrt[3]{-27} = -3$, because $-27 = (-3)^3$. The cube root of a negative number is always negative.

The properties of cube roots are similar to the corresponding properties of square roots.

Algebraic Properties of the Cube Root		
I	$\sqrt[3]{A}\sqrt[3]{B} = \sqrt[3]{AB}$	"The product of radicals is the radical of the product."
II	$\dfrac{\sqrt[3]{A}}{\sqrt[3]{B}} = \sqrt[3]{\dfrac{A}{B}}, \quad B \neq 0$	"The quotient of radicals is the radical of the quotient."
III	$\sqrt[3]{A^3} = \sqrt[3]{A}\sqrt[3]{A}\sqrt[3]{A} = \left(\sqrt[3]{A}\right)^3 = A$	"Cubing a cube root eliminates the radical."

IMPORTANT!
Just as taking square roots divides exponents by two, taking a cube root divides exponents by three. We will use this fact to simplify cube roots involving integers.

EXAMPLE H

Simplify: $\sqrt[3]{216}$.

First, we calculate the prime factorization of 216:

$$216 = 8 \cdot 27 = 2^3 \cdot 3^3$$

Then,

$$\sqrt[3]{216} = \sqrt[3]{2^3 \cdot 3^3} \overset{\text{II}}{=} \sqrt[3]{2^3} \cdot \sqrt[3]{3^3} \overset{\text{III}}{=} 2 \cdot 3 = 6 .$$

Notice how the properties of cube roots were used to simplify the radical in our last example, and how similar the process is to the process of simplifying square roots.

EXAMPLE I

Simplify: $\sqrt[3]{-864}$.

The cube root of a negative is the negative of the cube root:

$$\sqrt[3]{-864} = -\sqrt[3]{864} .$$

Next, we need to find the prime factorization of 864:

$$864 = 2 \cdot 432 = 2 \cdot 2 \cdot 216 = 2^2 \cdot 216 = 2^2 \cdot 2 \cdot 108$$

$$= 2^3 \cdot 108 = 2^3 \cdot 2 \cdot 54 = 2^4 \cdot 54 = 2^4 \cdot 2 \cdot 27$$

$$= 2^5 \cdot 27 = 2^5 \cdot 3^3$$

$$864 = 2^5 \cdot 3^3$$

So, $\sqrt[3]{-864} = -\sqrt[3]{2^5 \cdot 3^3}$. Now we divide each exponent in the radicand by 3:

Three goes into five once with a remainder of two.

$$= -\sqrt[3]{2^5 \cdot 3^3} = -2^1 \cdot \sqrt[3]{2^2 \cdot 3^3} = -2 \cdot \sqrt[3]{4 \cdot 3^3}$$

Three goes into three once with no remainder.

$$= -2 \cdot \sqrt[3]{4 \cdot 3^3} = -2 \cdot 3^1 \cdot \sqrt[3]{4} = \boxed{-6\sqrt[3]{4}}$$

Note:

- As with square roots, when there is no remainder (a remainder of zero), then nothing of that factor remains under the radical.

Extended Example 7a

Simplify: $\sqrt[3]{-200}$.

Extended Example 8a

Simplify: $\sqrt[3]{\dfrac{64x^{23}}{125y^{30}}}$.

MULTIPLYING AND DIVIDING RADICAL EXPRESSIONS

The concepts of square and cube roots can be generalized. If n is a natural number, we say the n^{th} root of A equals B, written

$$\sqrt[n]{A} = B \text{, whenever it is true that } A = B^n.$$

For example,

$$\sqrt[5]{32} = 2 \text{, because } 32 = 2^5.$$

All even roots are like the square root, in that they do not work with negative radicands.

All odd roots are like the cube root, in that the odd root of a negative number is negative.

Algebraic Properties of the n^{th} Root		
I	$\sqrt[n]{A}\sqrt[n]{B} = \sqrt[n]{AB}$	"The product of radicals is the radical of the product."
II	$\dfrac{\sqrt[n]{A}}{\sqrt[n]{B}} = \sqrt[n]{\dfrac{A}{B}}$, $B \neq 0$	"The quotient of radicals is the radical of the quotient."
III	$\sqrt[n]{A^n} = \left(\sqrt[n]{A}\right)^n = A$	"Raising an n^{th} root to the n^{th} power eliminates the radical."

IMPORTANT! Just as taking square roots divides exponents by two, and taking cube roots divides exponents by three, taking n^{th} roots divides exponents by n. If $n = 2$, the above table describes square roots. If $n = 3$, the above table describes cube roots. So the above table can be used for any number n.

MULTIPLYING AND DIVIDING RADICAL EXPRESSIONS

Extended Example 9a

Simplify: $\sqrt[10]{\dfrac{x^{73}y^{12}}{z^{90}}}$.

END OF LESSON

Simplify (assume non-negative variables): $\frac{\sqrt{252}}{\sqrt{112}}$

Simplify (assume non-negative variables): $\sqrt{35w^7v^3}\,\sqrt{70w^9v^{11}}$

Simplify and rationalize the denominator (assume non-negative variables): $\frac{14w^2\sqrt{15u}}{\sqrt{21w^3u}}$

Simplify.

$$\sqrt[5]{\frac{A^7B^{28}}{C^{15}}}$$

ADDING AND SUBTRACTING RADICAL EXPRESSIONS

Introduction

Adding and subtracting radical expressions requires first simplifying them and then combining like terms.

Review of Like Terms

To be "like terms," radicals have to be absolutely identical. As before, adding or subtracting like terms can be done using the distributive property:

$$2xy^2\sqrt{7xy} + 3xy^2\sqrt{7xy} = (2+3)\,xy^2\sqrt{7xy} = 5xy^2\sqrt{7xy}$$

When combining like terms in the usual way,

$$2xy^2\sqrt{7xy} + 3xy^2\sqrt{7xy} = 5xy^2\sqrt{7xy},$$

it is important to remember that the distributive law is needed to complete the process.

A Word to the Wise

Although it is true that $\sqrt{A}\sqrt{B} = \sqrt{AB}$, when adding, $\sqrt{A} + \sqrt{B} \neq \sqrt{A+B}$. Students often miss problems by mistakenly applying this rule.

ADDING AND SUBTRACTING RADICAL EXPRESSIONS

EXAMPLE A

Simplify: $4\sqrt{50} - 5\sqrt{32}$.

First, factor the radicands:

$$4\sqrt{50} - 5\sqrt{32} = 4\sqrt{2\cdot 5^2} - 5\sqrt{2^5}$$

Simplify each radical by dividing each exponent in each radicand by two:

Two goes into one zero times with a remainder of one, and

two goes into two once with no remainder:

$$= 4\sqrt[2]{2^1\cdot 5^2} - 5\sqrt{2^5} = 4\cdot 2^0\cdot 5^1\cdot\sqrt{2^1} - 5\sqrt{2^5}$$

$$= 4\cdot 1\cdot 5\cdot\sqrt{2} - 5\sqrt{2^5} = 20\sqrt{2} - 5\sqrt{2^5}$$

Two goes into five twice with a remainder of one:

$$= 20\sqrt{2} - 5\sqrt[2]{2^5} = 20\sqrt{2} - 5\cdot 2^2\sqrt{2^1} = 20\sqrt{2} - 20\sqrt{2} = 0$$

Note:

- Recall that anything raised to the 0 power equals 1. From here on, we will leave out factors involving numbers or variables with a 0 power, since multiplying by 1 does not change an answer or expression.

ADDING AND SUBTRACTING RADICAL EXPRESSIONS

Extended Example 1a

Simplify: $7\sqrt{63} - 3\sqrt{28}$.

ADDING AND SUBTRACTING RADICAL EXPRESSIONS

Note:
In this section of the course, all variables represent non-negative quantities.

EXAMPLE B

Simplify $3\sqrt{18x^3} + 5\sqrt{8x^5}$.

First, factor the coefficients in the radicands:

$$3\sqrt{18x^3} + 5\sqrt{8x^5} = 3\sqrt{2 \cdot 3^2 \cdot x^5} + 5\sqrt{2^3 \cdot x^5}$$

Simplify each radical by dividing each exponent in each radicand by two:

$$3\sqrt{2 \cdot 3^2 \cdot x^5} + 5\sqrt{2^3 \cdot x^5}$$

First we'll simplify the radical on the left:

Two goes into one zero times with a remainder of one, and

two goes into two once with no remainder, and

two goes into five twice with a remainder of one:

$$= 3\sqrt[2]{2^1 \cdot 3^2 \cdot x^5} + 5\sqrt{2^3x^5} = 3 \cdot 3^1 \cdot x^2 \cdot \sqrt{2^1 \cdot x^1} + 5\sqrt{2^3x^5}$$

$$= 9x^2\sqrt{2x} + 5\sqrt{2^3x^5}$$

continued...

Example B, continued...

Now we'll simplify the radical on the right:

Two goes into three once with a remainder of one, &

two goes into five twice with a remainder of one :

$$= 9x^2\sqrt{2x} + 5\sqrt[2]{2^3 \cdot x^5} = 9x^2\sqrt{2x} + 5 \cdot 2^1 \cdot x^2 \cdot \sqrt{2^1 \cdot x^1}$$

$$= 9x^2\sqrt{2x} + 10x^2\sqrt{2x}$$

Finally, we can combine like terms:

$$= 9x^2\sqrt{2x} + 10x^2\sqrt{2x}$$

$$= 19x^2\sqrt{2x}$$

Extended Example 2a

Simplify: $2z\sqrt{27z^5} - 7\sqrt{12z^7}$.

ADDING AND SUBTRACTING RADICAL EXPRESSIONS

EXAMPLE C

Simplify: $\sqrt[3]{24}\sqrt[3]{5}+\sqrt[3]{40}\sqrt[3]{3}$.

Start by factoring the radicands:

$$\sqrt[3]{24}\sqrt[3]{5}+\sqrt[3]{40}\sqrt[3]{3}=\sqrt[3]{2^3\cdot 3\cdot 5}+\sqrt[3]{2^3\cdot 5\cdot 3}$$
$$=\sqrt[3]{2^3\cdot 3\cdot 5}+\sqrt[3]{2^3\cdot 3\cdot 5}$$

Next, divide each exponent by 3. Notice that only the cubed factors are removed from the radical, since they are no longer cubed after having their exponents divided by 3:

$$=\sqrt[3]{2^3\cdot 3^1\cdot 5^1}+\sqrt[3]{2^3\cdot 3^1\cdot 5^1}=2^1\cdot\sqrt[3]{2^1\cdot 3^1\cdot 5^1}+2^1\cdot\sqrt[3]{2^1\cdot 3^1\cdot 5^1}$$
$$=2\sqrt[3]{30}+2\sqrt[3]{30}$$

Finally, combine like terms:

$$=2\sqrt[3]{30}+2\sqrt[3]{30}=4\sqrt[3]{30}$$

Extended Example 3a

Simplify: $\sqrt[3]{17000} - \sqrt[3]{1088}$.

EXAMPLE D

Simplify: $6x^2\sqrt[3]{2^3x^{14}y^{16}}+2y\sqrt[3]{3^3x^{20}y^{13}}$.

As usual, we start by factoring the coefficients of the radicands:

$$=6x^2\sqrt[3]{2^3x^{14}y^{16}}+2y\sqrt[3]{3^3x^{20}y^{13}}$$

Next, divide each exponent in the radicands by 3:

$$=6x^2\sqrt[3]{2^3x^{14}y^{16}}+2y\sqrt[3]{3^3x^{20}y^{13}}$$
$$=6x^2\cdot 2^1\cdot x^4\cdot y^5\cdot\sqrt[3]{x^2y^1}+2y\cdot 3^1\cdot x^6\cdot y^4\cdot\sqrt[3]{x^2y^1}$$
$$=12x^6y^5\sqrt[3]{x^2y}+6x^6y^5\sqrt[3]{x^2y}$$

Lastly, combine like terms:

$$=12x^6y^5\sqrt[3]{x^2y}+6x^6y^5\sqrt[3]{x^2y}$$
$$=18x^6y^5\sqrt[3]{x^2y}$$

ADDING AND SUBTRACTING RADICAL EXPRESSIONS

Extended Example 4a

Simplify: $6\sqrt[3]{2t^{35}} - t^4\sqrt[3]{250t^{23}}$.

ADDING AND SUBTRACTING RADICAL EXPRESSIONS

Rationalizing Denominators (Part 2)

Recall that "rationalizing the denominator" means removing all the radicals in the denominator. The technique we discussed in Section 1 does not work with some expressions involving square roots. For example,

$$\frac{1}{1+\sqrt{2}}.$$

If we try to multiply the numerator and denominator by the square root of two, it only eliminates one radical in the denominator while creating another radical. This method does not work.

An approach that does work involves a fact that you have used before:

$$(A+B)(A-B) = A^2 - B^2$$

This formula can help solve our current problem—we need to remove the radical from the denominator of $\frac{1}{1+\sqrt{2}}$. If we multiply the denominator by $1-\sqrt{2}$, we get:

$$\left(1+\sqrt{2}\right)\left(1-\sqrt{2}\right) = 1^2 - \left(\sqrt{2}\right)^2 = 1-2 = -1.$$

As you can see, the radicals are removed by this process. Expressions like $1+\sqrt{2}$ and $1-\sqrt{2}$ are called **conjugates**. Conjugates are expressions of the form $a+b$ and $a-b$.

So if we multiply the numerator and denominator by the conjugate of the denominator, it eliminates square roots from the denominator.

ADDING AND SUBTRACTING RADICAL EXPRESSIONS

EXAMPLE E

Rationalize the denominator: $\dfrac{1}{1+\sqrt{2}}$.

Multiply the numerator and denominator by the conjugate of the denominator:

$$= \frac{1}{1+\sqrt{2}} \cdot \frac{1-\sqrt{2}}{1-\sqrt{2}} = \frac{1-\sqrt{2}}{1^2-\left(\sqrt{2}\right)^2}$$

$$= \frac{1-\sqrt{2}}{1-2} = \frac{1-\sqrt{2}}{-1}$$

$$= \frac{\left(1-\sqrt{2}\right)\cdot(-1)}{(-1)\cdot(-1)}$$

$$= \frac{-1+\sqrt{2}}{1} = -1+\sqrt{2}$$

In this last example, by eliminating the radical in the denominator we ended up eliminating the entire denominator.

Extended Example 5a

Rationalize the denominator: $\frac{\sqrt{3}}{2+\sqrt{3}}$.

ADDING AND SUBTRACTING RADICAL EXPRESSIONS

EXAMPLE F

Rationalize the denominator: $\dfrac{1+\sqrt{y}}{1-\sqrt{y}}$.

Multiply the numerator and denominator by the conjugate of the denominator:

$$\frac{1+\sqrt{y}}{1-\sqrt{y}} = \frac{\left(1+\sqrt{y}\right)\cdot\left(1+\sqrt{y}\right)}{\left(1-\sqrt{y}\right)\cdot\left(1+\sqrt{y}\right)}$$

Multiply the numerator using the FOIL method. Use $(A+B)(A-B) = A^2 - B^2$ to simplify the denominator:

$$= \frac{\left(1+\sqrt{y}\right)\left(1+\sqrt{y}\right)}{\left(1-\sqrt{y}\right)\left(1+\sqrt{y}\right)} = \frac{1^2+\sqrt{y}+\sqrt{y}+\left(\sqrt{y}\right)^2}{1^2-\left(\sqrt{y}\right)^2}$$

$$= \frac{1+2\sqrt{y}+y}{1-y} \quad \text{or} \quad = \frac{1+y+2\sqrt{y}}{1-y}$$

15 of 16

Extended Example 6a

Rationalize the denominator: $\dfrac{\sqrt{uv}}{\sqrt{u}-\sqrt{v}}$.

END OF LESSON

Simplify (assume non-negative variables):

$$6AB\sqrt{20A^2B^6}+2B^4\sqrt{45A^4}$$

Simplify.

$$5u^2\sqrt{24u^3v^5}-2v\sqrt{54u^7v^3}$$

Simplify (assume non-negative variables):

$$3x^2 \sqrt[3]{125x^{14}y^{17}} + y\sqrt[3]{64x^{20}y^{14}}$$

Rationalize the denominator (assume non-negative variables):

$$\frac{\sqrt{A}}{\sqrt{A} + \sqrt{B}}$$

RADICAL EQUATIONS

Introduction

The main technique used to solve equations involving square roots, is to square both sides of the equal sign. Consider the radical property, $\left(\sqrt{A}\right)^2 = A$.

Squaring a square root eliminates the radical. Our first examples shows how to isolate the radical term on one side of the equal sign, and then square both sides of the equation to eliminate the radical.

EXAMPLE A

Solve: $\sqrt{x} = 3$.

Square both sides of the equal sign:

$$\left(\sqrt{x}\right)^2 = 3^2 \rightarrow \boxed{x = 9}$$

Checking this possible solution in the original equation, we get:

$$\sqrt{x} = 3 \rightarrow \sqrt{9} \overset{\text{Yes!}}{=} 3$$

So the solution is $x = 9$.

It's very important to remember to check your solutions whenever solving equations involving radicals. When you square both sides of an equation, the result could be one that does not make the original equation true. Such results are called extraneous solutions. You should ALWAYS check to see that your proposed solution really does solve the original equation.

RADICAL EQUATIONS

Extended Example 1a

Solve: $\sqrt{2x} = 4$.

EXAMPLE B

Solve: $\sqrt{2x-5}-4=1$.

This time, the radical is not by itself. We must first isolate the radical on one side of the equal sign:

$$\begin{array}{rcl} \sqrt{2x-5}-4 &=& 1 \\ +4 && +4 \\ \hline \sqrt{2x-5} &=& 5 \end{array}$$

Now we can square both sides:

$$\left(\sqrt{2x-5}\right)^2 = 5^2$$

$$\begin{array}{rcl} 2x-5 &=& 25 \\ +5 && +5 \\ \hline 2x &=& 30 \end{array}$$

$$\frac{\cancel{2}x}{\cancel{2}} = \frac{30}{2}$$

$$\boxed{x = 15}$$

Finally, check to see that we have found a solution to the original equation:

$$\sqrt{2x-5}-4=1 \;\rightarrow\; \sqrt{2\cdot 15-5}-4 \overset{?}{=} 1 \;\rightarrow\; \sqrt{30-5}-4 \overset{?}{=} 1$$

$$\rightarrow\; \sqrt{25}-4 \overset{?}{=} 1 \;\rightarrow\; 5-4 \overset{?}{=} 1$$

$$\rightarrow\; 1 \overset{\text{Yes!}}{=} 1$$

So the solution is $x = 15$.

Extended Example 2a

Solve: $\sqrt{6x-5}-4=3$.

RADICAL EQUATIONS

EXAMPLE C

Solve: $\sqrt{3x-1}-\sqrt{2x+3}=0$.

This time we want to place the radicals on opposite sides of the equal sign. Then, squaring both sides of the equation will eliminate both radicals:

$$\begin{aligned} \sqrt{3x-1}-\sqrt{2x+3} &= 0 \\ +\sqrt{2x+3} \quad & \quad +\sqrt{2x+3} \\ \hline \sqrt{3x-1} &= \sqrt{2x+3} \end{aligned}$$

Now, square both sides:

$$\begin{aligned} \left(\sqrt{3x-1}\right)^2 &= \left(\sqrt{2x+3}\right)^2 \\ 3x-1 &= 2x+3 \\ -2x \quad & \quad -2x \\ \hline x-1 &= 3 \\ +1 \quad & \quad +1 \\ \hline \boxed{x = 4} \end{aligned}$$

Check the solution:

$$\begin{aligned} \sqrt{3x-1}-\sqrt{2x+3}=0 \quad &\rightarrow \quad \sqrt{3\cdot 4-1}-\sqrt{2\cdot 4+3} \stackrel{?}{=} 0 \\ &\rightarrow \quad \sqrt{12-1}-\sqrt{8+3} \stackrel{?}{=} 0 \\ &\rightarrow \quad \sqrt{11}-\sqrt{11} \stackrel{\text{Yes!}}{=} 0 \end{aligned}$$

Once again, our solution checks out.

EXAMPLE D

Solve: $\sqrt{10x-13}=-6$.

This equation has no solution. The left side of the equal sign is not negative while the right side is negative. No quantity can be both negative and not negative at the same time! Thus no solution is possible.

If you hadn't noticed this at first, and instead went through the process of solving the equation by squaring both sides, then eventually you would find a possible solution. However, when you checked your answer, you would find that it did not make the original equation true.

It's better in cases like this to simply notice that no solution is possible. By observing that the left side of the equal sign is positive (or zero) while the right side is negative, you know that the two sides cannot possibly be equal.

Extended Example 3a

Solve: $\sqrt{12-3x}+14=13$.

EXAMPLE E

Solve: $\sqrt{x^2+5}-x=2$.

First, isolate the radical on one side of the equal sign. Add x to both sides:

$$\sqrt{x^2+5}=x+2$$

Then square both sides. Use the binomial squared formula, $(A+B)^2=A^2+2AB+B^2$ for the right side of the equal sign:

$$\left(\sqrt{x^2+5}\right)^2=(x+2)^2$$
$$x^2+5=x^2+4x+4$$

Subtract x^2 from both sides of the equation and then solve for x:

$$5=4x+4$$
$$1=4x$$
$$\boxed{\frac{1}{4}=x}$$

continued...

RADICAL EQUATIONS

Example E, continued...

Lastly we check our possible solution:

$$\sqrt{x^2+5}-x=2 \rightarrow \sqrt{\left(\frac{1}{4}\right)^2+5}-\left(\frac{1}{4}\right) \stackrel{?}{=} 2$$

$$\rightarrow \sqrt{\frac{1}{16}+5}-\frac{1}{4} \stackrel{?}{=} 2$$

$$\rightarrow \sqrt{\frac{1}{16}+\frac{5\cdot 16}{16}}-\frac{1}{4} \stackrel{?}{=} 2$$

$$\rightarrow \sqrt{\frac{1+80}{16}}-\frac{1}{4} \stackrel{?}{=} 2$$

$$\rightarrow \sqrt{\frac{81}{16}}-\frac{1}{4} \stackrel{?}{=} 2$$

$$\rightarrow \frac{9}{4}-\frac{1}{4} \stackrel{?}{=} 2$$

$$\rightarrow \frac{8}{4} \stackrel{\text{Yes!}}{=} 2$$

Our solution, $x=\frac{1}{4}$, checks out.

Extended Example 4a

Solve: $x - \sqrt{x^2 - 1} = 2$.

EXAMPLE F

Solve: $\sqrt{5x-1}+3=x$.

Start by isolating the radical on one side of the equal sign by subtracting 3 from both sides:

$$\begin{array}{r} \sqrt{5x-1}+3 = x \\ \underline{-3 \quad -3} \\ \sqrt{5x-1} = x-3 \end{array}$$

Square both sides of the equal sign:

$$\left(\sqrt{5x-1}\right)^2 = (x-3)^2$$

$$5x-1 = x^2-6x+9$$

Solve for x . Since this is a quadratic equation, we will have to move everything on one side of the equal sign and set it equal to 0. Then we can factor the quadratic to solve the equation. To move everything on one side of the equal sign, subtract $5x$ from both sides and add 1 to both sides:

$$\begin{array}{r} 5x-1 = x^2-6x+9 \\ \underline{-5x+1 \qquad -5x+1} \\ 0 = x^2-11x+10 \end{array}$$

continued...

Example F, continued...

Factor the right side to solve for x:

$$0 = x^2 - 11x + 10$$
$$0 = (x - 10) \cdot (x - 1)$$
$$\downarrow \qquad \downarrow$$
$$\boxed{x = 10} \text{ or } \boxed{x = 1}$$

Finally, check to see if these possible solutions work by trying them in the original equation:

$\boxed{x = 10}$: $\sqrt{5x - 1} + 3 = x \rightarrow \sqrt{5 \cdot 10 - 1} + 3 \overset{?}{=} 10$

$\rightarrow \sqrt{50 - 1} + 3 \overset{?}{=} 10 \rightarrow \sqrt{49} + 3 \overset{?}{=} 10 \rightarrow 7 + 3 \overset{\text{Yes!}}{=} 10$

$\boxed{x = 1}$: $\sqrt{5x - 1} + 3 = x \rightarrow \sqrt{5 \cdot 1 - 1} + 3 \overset{?}{=} 1$

$\rightarrow \sqrt{5 - 1} + 3 \overset{?}{=} 1 \rightarrow \sqrt{4} + 3 \overset{?}{=} 1 \rightarrow 2 + 3 \overset{\text{No!}}{=} 1$

Of the two possible solutions we found, only one of them really solves the equation: $x = 10$. The other was an extraneous solution.

Extended Example 5a

Solve: $\sqrt{2x}+4=x$.

END OF LESSON

Solve: $3\sqrt{3x+1} = 2\sqrt{2x-1}$

Solve.

$\sqrt{7x-5} - \sqrt{8x+2} = 0$

Solve: $\sqrt{3x-8} = \sqrt{8x-3}$

Solve: $\sqrt{4x+7} + 2\sqrt{x} = 7$

Solve.

$x - \sqrt{2x^2 + 50} = 5$

Solve.

$\sqrt{3x} + 2 = 3x$

COMPLEX NUMBERS

Introduction

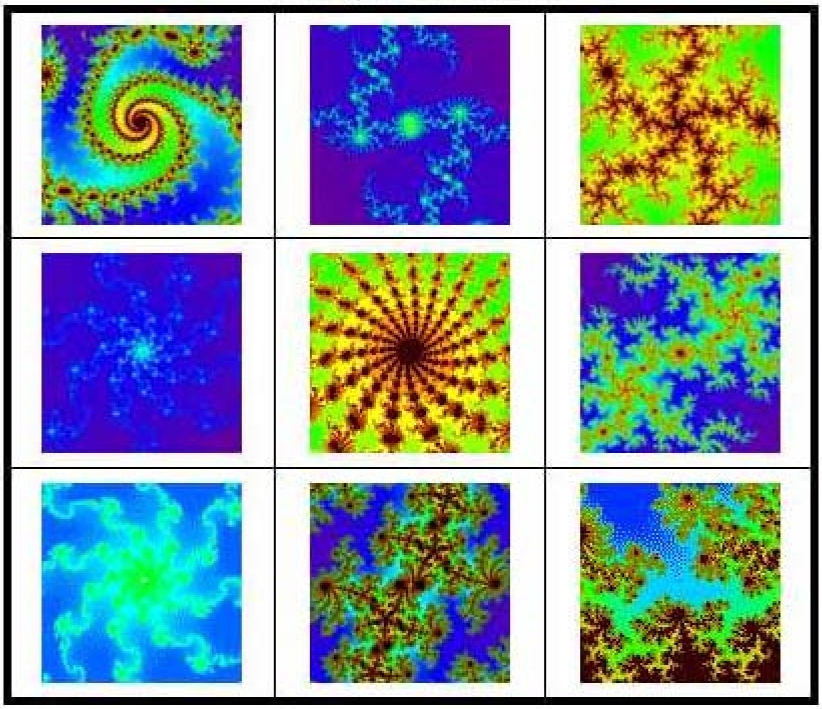
In this section, you will be introduced to a new value that is represented by the single letter, i . This value, known as the **imaginary number** i , is extremely useful in solving real-world applications including modeling weather systems, describing the flow of fluids and electricity, designing the cross-section of airplane wings, creating special effects in movies, and more.

COMPLEX NUMBERS

Fractal Art

One interesting and fun application of the imaginary number i is the generation of fractal art. Some examples are shown below. You can do a web-based search on "fractal art" for many more examples.

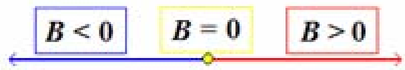
Examples of Fractals



© 2006 Rafael Espericueta

COMPLEX NUMBERS

We have seen that squaring a real number cannot result in a negative quantity. Suppose B is a real number (any point on the number line). Then there are three distinct possibilities:



1. B is positive, $B > 0$: Since a positive multiplied by a positive is positive, B^2 is positive: $B^2 > 0$.

   For example, $2^2 = 2 \cdot 2 = 4$ is positive.

2. B is zero, $B = 0$: Since zero times zero is zero, B^2 is zero: $B^2 = 0$.

   For example, $0^2 = 0 \cdot 0 = 0$.

3. B is negative, $B < 0$: Since a negative times a negative is positive, B^2 is positive: $B^2 > 0$.

   For example, $(-5)^2 = (-5) \cdot (-5) = 25$ is positive.

COMPLEX NUMBERS

In no case is B^2 negative. Since $\sqrt{A} = B$ if and only if $A = B^2$, notice what happens with negatives under square roots (and all even roots). Consider, for example, the case where A is negative one:

$$\sqrt{-1} = B \quad \text{if and only if} \quad -1 = B^2.$$

No real number has a square equal to negative one, so B cannot be a real number. There are other kinds of numbers besides real numbers, though. One of these numbers is written as i, and it has the property that

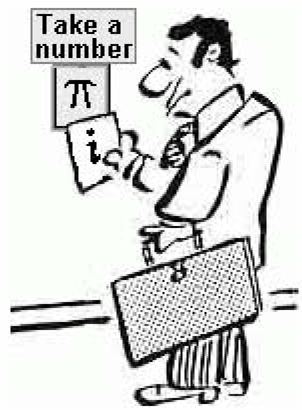
$$i^2 = -1.$$

The letter i stands for "imaginary."

What is reality?

Though called imaginary,
the imaginary numbers are just as real
as the real numbers.

Though called real,
the real numbers are just as imaginary
as the imaginary numbers.



© 2006 Rafael Espericueta

COMPLEX NUMBERS

Since $i^2 = -1$, we can also write $i = \sqrt{-1}$. This new number can be multiplied and even added to real numbers. When you multiply a real number times i, the result is an imaginary number. For example,

$$i,\ 2i,\ \frac{2}{7}i,\ -5i,\ -\pi i,\ \sqrt{2}\,i$$

are examples of imaginary numbers.

When you add or subtract a real number and an imaginary number, the result is a **complex number**. For example,

$$1+i,\ 2i,\ 1-3i,\ \frac{3}{10}-\frac{2}{7}i,\ \sqrt{3}+\pi i,\ -\sqrt{3}-\sqrt{2}\,i$$

are examples of complex numbers. The real number in such a sum is called the **real part** of the complex number, and the coefficient of i is called the **imaginary part** of the complex number. The imaginary numbers themselves are complex numbers with the real part equal to zero, while the real numbers are complex numbers with the imaginary part equal to zero.

To do algebra with complex numbers, treat the i as if it were any variable. Combine like terms as usual.

Remember that whenever i is squared, you may replace it with -1 since $i^2 = -1$.

COMPLEX NUMBERS

EXAMPLE A

Add the complex numbers: $(5-3i)+(2+9i)$.

Add these expressions as you would if i were an ordinary variable, like x:

$$= 5-3i+2+9i$$

$$= 5-3i+2+9i = 7+6i$$

The parentheses were not needed. They were included only to emphasize that we were adding two complex numbers.

EXAMPLE B

Subtract the complex numbers: $(7-2i)-(4+9i)$.

Subtract these expressions as you would if i were an ordinary variable, like x. Distribute the negative to eliminate parentheses:

$$(7-2i)-(4+9i) = 7-2i-4-9i$$

$$= 7-2i-4-9i$$

$$= 3-11i$$

COMPLEX NUMBERS

Extended Example 1a

Add the complex numbers $(12-6i)+(-4+3i)$.

COMPLEX NUMBERS

EXAMPLE C

Multiply the complex numbers: $(5-3i)(2+7i)$.

Use the FOIL method and combine like terms:

$$
\begin{aligned}
(5-3i)(2+7i) &= 5\cdot 2 + 5\cdot 7i - 3i\cdot 2 - 3i\cdot 7i \\
&= 10 + 35i - 6i - 21i^2 \\
&= 10 + 29i - 21i^2
\end{aligned}
$$

Now use that most important property of i, that $i^2 = -1$.

$$
\begin{aligned}
= 10 + 29i - 21i^2 &= 10 + 29i - 21\cdot(-1) \\
&= 10 + 29i + 21 \\
&= 31 + 29i
\end{aligned}
$$

COMPLEX NUMBERS

Extended Example 2a

Multiply the complex numbers $(1-2i)(7-3i)$.

COMPLEX NUMBERS

The real numbers become geometrical once we visualize them as the points on a number line. There is also a way to think of the complex numbers geometrically. Earlier we stated that:

> The real numbers are those complex numbers that have the imaginary part equal to zero, just as the imaginary numbers are those complex numbers that have the real part equal to zero.

This reminds us of a similar fact concerning points on the coordinate axes:

> The points on the x axis are those points in the plane that have y-coordinate equal to zero, just as the points on the y axis are those points in the plane that have x-coordinate equal to zero.

Think of the real number line as the x axis, then the imaginary numbers can be thought of as the points on the y axis. The complex numbers are then all the points in the plane. Geometrically, the complex number $a+bi$ is the point (a,b).

COMPLEX NUMBERS

The number 1 is the point $(1,0)$ on the x axis. To see why, consider:

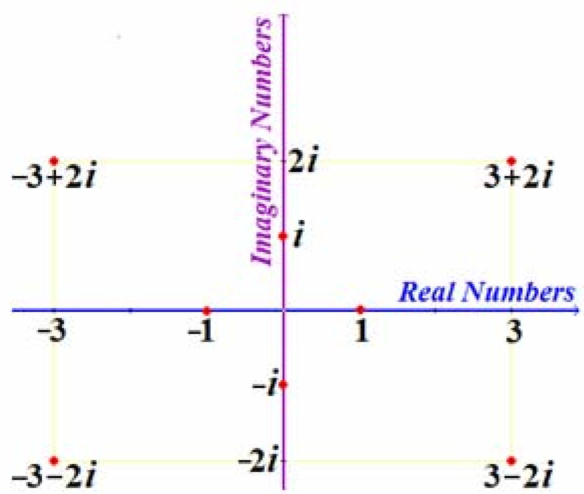
$$i = 1 + 0 \cdot i = (1,0) = (1,0).$$

The number i is the point $(0,1)$ on the y axis. This is because:

$$i = 0 + 1 \cdot i = (0,1) = (0,1).$$

Complex number	Cartesian point
1	$(1,0)$
i	$(0,1)$
-1	$(-1,0)$
$-i$	$(0,-1)$
$3+2i$	$(3,2)$
$-3+2i$	$(-3,2)$
$-3-2i$	$(-3,-2)$
$3-2i$	$(3,-2)$

The Complex Plane



Since $i^2 = -1$, we can write $i = \sqrt{-1}$. This gives us a way to rewrite square root radicals with negative radicands as imaginary numbers. Now such expressions have a concrete meaning; they are the points on the plane.

COMPLEX NUMBERS

EXAMPLE D

Rewrite as an imaginary number: $\sqrt{-9}$.

Follow these simple steps:

$$\sqrt{-9} = \sqrt{-1 \cdot 9} = \sqrt{-1} \cdot \sqrt{9} = i \cdot 3 = 3i$$

EXAMPLE E

Rewrite as an imaginary number: $\sqrt{-17}$.

$$\sqrt{-17} = \sqrt{-1 \cdot 17} = \sqrt{17} \cdot \sqrt{-1} = \sqrt{17} \cdot i = \sqrt{17}\, i$$

Note:

- All you need to do is erase the negative under the radicand and put an i outside the radical.

COMPLEX NUMBERS

Question: Rewrite as an imaginary number: $\sqrt{-23}$.

Question: Rewrite the radical as an imaginary number: $-\sqrt{-36}$.

Question: Rewrite the radical as an imaginary number: $-\sqrt{-40}$.

COMPLEX NUMBERS

EXAMPLE F

Rewrite the expression as a complex number: $4+\sqrt{-18}$.

Replace the negative under the radical with an i outside the radical:

$$4+\sqrt{-18} = 4+\sqrt{18}\,i$$

Factor the radicand and simplify the radical:

$$= 4+\sqrt{9\cdot 2}\,i = 4+\sqrt{9}\cdot\sqrt{2}\cdot i$$

$$= 4+3\cdot\sqrt{2}\cdot i = 4+3\sqrt{2}\,i$$

Extended Example 3a

Rewrite the expression as a complex number: $7+\sqrt{-60}$.

Simplify.

$(-11+6i)-(-6-9i)$

Simplify.

$(3-11i)(2+3i)$

Write the given expression as a complex number.

$-\sqrt{-19}$

RATIONAL EXPONENTS

Introduction

As we have seen, radicals involve dividing exponents. So perhaps it is not too surprising that they can be expressed using fractional exponents. In fact, since taking an n^{th} root divides exponents by n, it must be true that

$$\sqrt[n]{a^m} = a^{\frac{m}{n}}.$$

The algebraic rules for dealing with exponents also apply when the exponents are fractions. Converting radicals to fractional exponents often makes calculations with radicals much easier, since you may then apply the exponent properties.

RATIONAL EXPONENTS

If $m = 1$, $\sqrt[n]{a^m} = a^{\frac{m}{n}}$ becomes

$$\sqrt[n]{a} = a^{\frac{1}{n}}.$$

The formula above allows us to numerically compute radicals on any calculator capable of raising a number to a power.

EXAMPLE A

Write each radical as a fractional exponent: $\sqrt{3}$, $\sqrt[8]{6^3}$, $\sqrt[3]{7^2}$, $\sqrt[5]{11^7}$.

$$\sqrt{3} = \sqrt[2]{3^1} = 3^{\frac{1}{2}}, \quad \sqrt[8]{6^3} = 6^{\frac{3}{8}}, \quad \sqrt[3]{7^2} = 7^{\frac{2}{3}}, \quad \sqrt[5]{11^7} = 11^{\frac{7}{5}}$$

EXAMPLE B

Write each fractional exponent as a radical: $7^{\frac{1}{2}}$, $5^{\frac{8}{11}}$, $17^{\frac{4}{9}}$, $15^{\frac{7}{3}}$.

$$7^{\frac{1}{2}} = \sqrt[2]{7^1} = \sqrt{7}, \quad 5^{\frac{8}{11}} = \sqrt[11]{5^8}, \quad 17^{\frac{4}{9}} = \sqrt[9]{17^4}, \quad 15^{\frac{7}{3}} = \sqrt[3]{15^7}$$

Add this property to your knowledge of properties of exponents:

$$\sqrt[n]{a^m} = a^{\frac{m}{n}}.$$

RATIONAL EXPONENTS

You should review the properties of exponents below. For the remainder of this lesson, we will refer to the exponent properties using Roman numerals (see the following table). A Roman numeral above an equal sign in an example refers to the property that justifies that step.

Properties of Exponents	
I	$a^0 = 1$
II	$a^1 = a$
III	$a^m \cdot a^n = a^{m+n}$
IV	$\frac{a^m}{a^n} = a^{m-n}$
V	$\left(a^m\right)^n = a^{m \cdot n}$
VI	$(ab)^n = a^n \cdot b^n$
VII	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
VIII	$a^{-n} = \frac{1}{a^n}$
IX	$\sqrt[n]{a^m} = a^{\frac{m}{n}}$

EXAMPLE C

Simplify: $125^{\frac{2}{3}}$.

Begin by factoring the 125:

$$125^{\frac{2}{3}} = \left(5^3\right)^{\frac{2}{3}}$$

Now use property V, $\left(a^m\right)^n = a^{m \cdot n}$ (when raising a power to a power, multiply the exponents):

$$= \left(5^3\right)^{\frac{2}{3}} \overset{V}{=} 5^{3\cdot\left(\frac{2}{3}\right)}$$

$$= 5^{\frac{3\cdot 2}{3}} = 5^{\frac{\cancel{3}\cdot 2}{\cancel{3}}} = 5^2 = 25$$

Extended Example 1a

Simplify: $16^{\frac{5}{4}}$.

RATIONAL EXPONENTS

EXAMPLE D

Simplify: $216^{\frac{2}{3}}$.

This is the same problem as Extended Example 1c, solved in a different way.

Find the prime factorization of 216.

$$216 = 2 \cdot 108 = 2 \cdot 2 \cdot 54 = 2^2 \cdot 54 = 2^2 \cdot 2 \cdot 27 = 2^3 \cdot 3^3$$

So, $216^{\frac{2}{3}} = \left(2^3 \cdot 3^3\right)^{\frac{2}{3}}$.

Use exponent property VI, $(ab)^n = a^n \cdot b^n$, backwards on the expression in the parentheses.

$$= \left(2^3 \cdot 3^3\right)^{\frac{2}{3}} \overset{\text{VI}}{=} \left((2 \cdot 3)^3\right)^{\frac{2}{3}} = \left(6^3\right)^{\frac{2}{3}}$$

Use exponent property V, $\left(a^m\right)^n = a^{m \cdot n}$.

$$= \left(6^3\right)^{\frac{2}{3}} \overset{\text{V}}{=} 6^{3\left(\frac{2}{3}\right)} = 6^{\frac{\cancel{3} \cdot 2}{\cancel{3}}} = 6^2 = 36$$

RATIONAL EXPONENTS

EXAMPLE E

Simplify: $27^{-\frac{2}{3}}$.

First factor the 27:

$$27^{-\frac{2}{3}} = \left(3^3\right)^{-\frac{2}{3}}$$

Multiply the exponents (property V, $\left(a^m\right)^n = a^{m\cdot n}$):

$$= \left(3^3\right)^{-\frac{2}{3}} \overset{\text{V}}{=} 3^{3\cdot\left(-\frac{2}{3}\right)} = 3^{-\frac{3\cdot 2}{3}} = 3^{-\frac{\cancel{3}\cdot 2}{\cancel{3}}} = 3^{-2}$$

Use the definition of negative exponents (property VIII, $a^{-n} = \frac{1}{a^n}$):

$$3^{-2} \overset{\text{VIII}}{=} \frac{1}{3^2} = \frac{1}{9}$$

RATIONAL EXPONENTS

Extended Example 2a

Simplify: $16^{-\frac{3}{4}}$.

RATIONAL EXPONENTS

EXAMPLE F

Simplify: $\left(\frac{81}{16}\right)^{-\frac{3}{4}}$.

We start with property VII, $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$:

$$\left(\frac{81}{16}\right)^{-\frac{3}{4}} \overset{\text{VII}}{=} \frac{81^{-\frac{3}{4}}}{16^{-\frac{3}{4}}}$$

Next, factor the numbers in the parentheses and apply property V, $\left(a^m\right)^n = a^{m \cdot n}$:

$$= \frac{\left(3^4\right)^{-\frac{3}{4}}}{\left(2^4\right)^{-\frac{3}{4}}} \overset{\text{V}}{=} \frac{3^{4\cdot\left(-\frac{3}{4}\right)}}{2^{4\cdot\left(-\frac{3}{4}\right)}} = \frac{3^{-\frac{4\cdot 3}{4}}}{2^{-\frac{4\cdot 3}{4}}} = \frac{3^{-\frac{\cancel{4}\cdot 3}{\cancel{4}}}}{2^{-\frac{\cancel{4}\cdot 3}{\cancel{4}}}} = \frac{3^{-3}}{2^{-3}}$$

Lastly, use property VIII, $a^{-n} = \frac{1}{a^n}$, and simplify:

$$= \frac{3^{-3}}{2^{-3}} \overset{\text{VIII}}{=} \frac{2^3}{3^3} = \frac{8}{27}$$

RATIONAL EXPONENTS

EXAMPLE G

Simplify: $\left(\frac{81}{16}\right)^{-\frac{3}{4}}$.

This is the same problem as Example F, solved in a different way.

$$\left(\frac{81}{16}\right)^{-\frac{3}{4}} = \left(\frac{3^4}{2^4}\right)^{-\frac{3}{4}} \overset{\text{VII}}{=} \left(\left(\frac{3}{2}\right)^4\right)^{-\frac{3}{4}}$$

$$\overset{\text{V}}{=} \left(\frac{3}{2}\right)^{4\left(-\frac{3}{4}\right)} = \left(\frac{3}{2}\right)^{-\frac{4\cdot 3}{4}} = \left(\frac{3}{2}\right)^{-\frac{\cancel{4}\cdot 3}{\cancel{4}}} = \left(\frac{3}{2}\right)^{-3}$$

$$= \left(\frac{3}{2}\right)^{-3} \overset{\text{VIII}}{=} \frac{1}{\left(\frac{3}{2}\right)^3}$$

$$\overset{\text{VII}}{=} \frac{1}{\left(\frac{3^3}{2^3}\right)} = \frac{1}{\left(\frac{27}{8}\right)} = 1\cdot\left(\frac{8}{27}\right) = \frac{8}{27}$$

Notice that the same exponent properties were used in both solutions to this last problem, though the rules were used in a different order.

Extended Example 3a

Simplify: $\left(\frac{1}{49}\right)^{-\frac{1}{2}}$.

RATIONAL EXPONENTS

Some calculators only allow you to compute approximations of square roots. To compute higher roots (cube roots, fourth roots, etc.) you need to first write the radical in fractional exponent form. You will need a calculator that allows you to compute powers to do the next examples.

EXAMPLE H

Evaluate $\sqrt[5]{127}$. Round your answer to the nearest millionth.

Start by writing the expression in fractional exponent form:

$$\sqrt[5]{127} = \sqrt[5]{127^1} = 127^{\frac{1}{5}}$$

Try to compute this on your calculator. On some calculators, you can use the following calculator buttons (shown here as boxes):

$$127 \; \boxed{y^x} \; 5 \; \boxed{\frac{1}{x}} \; \boxed{=}$$

On calculators that use algebraic notation you should enter:

$$127 \; \boxed{\wedge} \; \boxed{(} \; 1 \; \boxed{/} \; 5 \; \boxed{)} \; \boxed{=}$$

continued...

Example H, continued...

On reverse-Polish calculators, enter:

$$127 \boxed{enter}\ 5 \boxed{\frac{1}{x}} \boxed{y^x}$$

Note: Some calculators have a $\boxed{x^y}$ button instead of $\boxed{y^x}$, but they function the same way.

Your answer should be something close to **2.63487941277060484854**, though probably with fewer digits.

The last step is to round this number to the nearest millionth:

$\cong \boxed{2.634879}$

Note: $\cong$ means approximately

It is very important that you know how to compute such radicals on your calculator. If none of the above methods seem to work on your calculator, refer to your calculator manual (if possible). Often such manuals include examples of the precise steps required for various calculations.

RATIONAL EXPONENTS

Extended Example 4a

Evaluate $\sqrt[3]{17}$. Round your answer to the nearest thousandth.

END OF LESSON

Simplify.

$$\left(\frac{16}{81}\right)^{-\frac{3}{4}}$$

Simplify.

$$\left(x^{9}y^{3}\right)^{-\frac{4}{3}}$$

Convert to fractional exponents, and then evaluate using a calculator. Round your answer to the nearest ten-thousandth.

$$\sqrt[10]{1,234,567}$$