GRAPHING LINEAR EQUATIONS IN TWO VARIABLES

Introduction

In this lesson, we will examine how to graph linear equations in two variables. As the word "linear" suggests, the graphs of all such equations are lines.

GRAPHING LINEAR EQUATIONS IN TWO VARIABLES

A Word to the Wise

You may want to review the lesson called "The Cartesian Coordinate Plane and the Distance Formula" before beginning this lesson. Links to some material from that lesson are provided on the next screen for your reference.

Let's consider the equation $y = 2x + 1$.

For any chosen value of x, we can find a corresponding value for y.

For example, let $x = 0$ in the equation above. Substituting the value $x = 0$ into the equation, we get:

$$y = 2 \cdot x + 1$$

$$y = 2 \cdot 0 + 1$$

$$y = 0 + 1$$

$$y = 1$$

Therefore, if $x = 0$, then $y = 1$. This means that if we substitute the values $x = 0$ and $y = 1$, the equation makes a true statement:

$$y = 2 \cdot x + 1$$

$$1 \stackrel{?}{=} 2 \cdot 0 + 1$$

$$1 \stackrel{?}{=} 0 + 1$$

$$1 = 1 \quad \text{True!}$$

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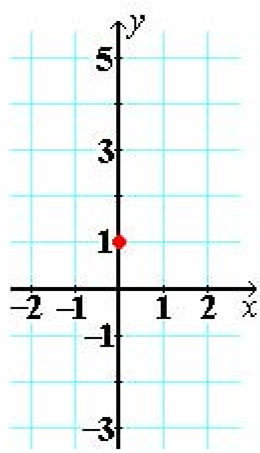
Because they make a true statement, the values $x = 0$ and $y = 1$ are said to **satisfy** the equation $y = 2x + 1$. These values are the x and y-coordinates of a point in the Cartesian coordinate plane—point $(0,1)$. To graph the equation $y = 2x + 1$ is to graph the line with ordered pairs that satisfy the equation. Most points do not satisfy this equation. Surprisingly, those points that do all lie on the same straight line when graphed.

We can now start our graph by plotting the one point that we know is on the line $y = 2x + 1$, the point $(0,1)$.



Finding only one more point will allow us to graph this line. However, it's a good idea to plot more than two points (especially as you're learning). If you find that they don't lie on the same line, you'll know you've made a mistake.

GRAPHING LINEAR EQUATIONS IN TWO VARIABLES

Let's find a few more points that satisfy $y = 2x + 1$, and then graph the line. It's helpful to make a table with columns for the x-coordinate, the y-coordinate, and the resulting point. Start by filling in a few x values. While you're free to pick <u>any</u> numbers for the x-column of this table (including numbers like $7/13$ or 3.14159 or $4,729$), it's a good idea to pick numbers that are easier to work with.)

For the table below, we have already found the first point, $(0,1)$. Next we pick other small integers to fill in the x-column:

x	y		Point
0	1	$\rightarrow$	$(0,1)$
1			
2			
-1			
-2			

We fill in the blanks in the y column by substituting each of the x values into the equation to find the corresponding y value. This substitution and the resulting points are shown on the next screen.

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The completed x-y table below shows our chosen x values and the y values that result when those x values are substituted into the original equation. The substitution for x to solve for y is shown to the left of the table. The resulting ordered pairs are listed on the right.

Original equation: $y = 2 \cdot x + 1$	x	y		**Point**
$y = 2 \cdot 0 + 1$ $y = 0 + 1$ $y = 1$	0	1	$\rightarrow$	$(0,1)$
$y = 2 \cdot 1 + 1$ $y = 2 + 1$ $y = 3$	1	3	$\rightarrow$	$(1,3)$
$y = 2 \cdot 2 + 1$ $y = 4 + 1$ $y = 5$	2	5	$\rightarrow$	$(2,5)$
$y = 2 \cdot (-1) + 1$ $y = -2 + 1$ $y = -1$	-1	-1	$\rightarrow$	$(-1,-1)$
$y = 2 \cdot (-2) + 1$ $y = -4 + 1$ $y = -3$	-2	-3	$\rightarrow$	$(-2,-3)$

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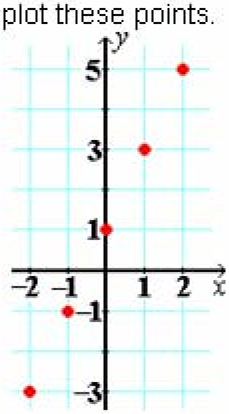
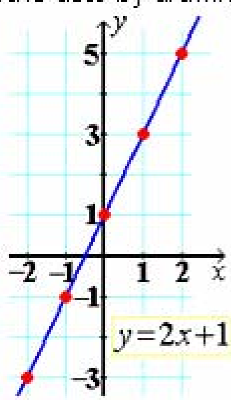
Now we have a table of x and y values (below), and their corresponding points, for the equation $y = 2x + 1$. Next, we plot these points.

x	y	Point
0	1	$(0,1)$
1	3	$(1,3)$
2	5	$(2,5)$
-1	-1	$(-1,-1)$
-2	-3	$(-2,-3)$

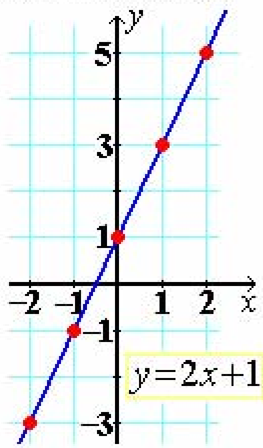


To graph the line, just connect the dots by drawing a line through the points:



GRAPHING LINEAR EQUATIONS IN TWO VARIABLES

Now you see why we call equations such as $y = 2x + 1$ **linear**. Their graphs are all lines. In this example, we showed only a small section of a line that extends infinitely in both directions. Sometimes arrows are shown on both ends of the line in a graph to indicate that the line is infinite.



Each point on a line represents a **solution** to its linear equation. If a point is on the line, the point's x-y coordinates make the equation a true statement. If a point is not on the line, its x-y coordinates make the equation a false statement. Plotting only two solutions to a linear equation would allow us to graph the line and visualize all of the infinitely many solutions, one for each point on the line.

However, although it only takes two points to graph a line, be SURE you have the correct points! Finding more than two points is one way to help make sure.

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Extended Example 1a

Graph the line $y = 3 - 4x$.

EXAMPLE A

Graph the line $2x + y = 3$.

This is a linear equation in **standard form**, $Ax + By = C$, with constants A, B, C. For this line, $A = 2$, $B = 1$, $C = 3$. Try to find the simplest numbers that make this equation true. Start with 0 and 1. Let $x = 0$ and find y. Again, we start with a table:

$2x + y = 3$	x	y		Point
$2 \cdot 0 + y = 3$				
$0 + y = 3$				
$y = 3$	0	3	$\rightarrow$	$(0,3)$

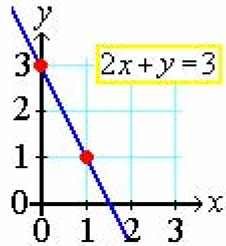
We found that if $x = 0$, then $y = 3$. We need one more point to plot this line. If we let $x = 1$, we get another easy solution:

$2x + y = 3$	x	y		Point
$2 \cdot 0 + 3 = 3$	0	3	$\rightarrow$	$(0,3)$
$2 \cdot 1 + y = 3$				
$2 + y = 3$				
$y = 1$	1	1	$\rightarrow$	$(1,1)$

continued...

Example A, continued…

If we plot $(0,3)$ and $(1,1)$ and draw the line through them, we've graphed the given line.



As was mentioned above, it only takes two points to graph a line. However, it doesn't hurt to find at least <u>three</u> points when graphing lines in order to rule out possible mistakes. If your three points aren't on a straight line, go back to the drawing board—there's a mistake in one of your calculations.

Note:

- In the previous examples, we've substituted numbers for x and solved to find corresponding values for y. Equally, one can find points on a line by substituting numbers for y and solving for x. There's a simple way to decide if you should start by substituting a number for x or for y: do whichever is easiest. If you can't decide, try it both ways!
- The most popular number to substitute with is 0; the second most popular number is 1. These are usually the easiest numbers to substitute into an algebraic expression because the arithmetic is easier to perform.

Extended Example 2a

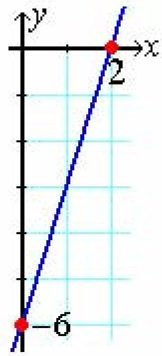
Graph the line $x - 3y = 1$.

Intercepts

The place where a line crosses the y-axis is called the y **-intercept**. Recall that a point's x-coordinate is essentially the distance from that point to the y-axis. So, all the points on the y-axis have an x-coordinate of 0.

Similarly, the place where a line crosses the x-axis is called the x **-intercept**. Recall that a point's y-coordinate is essentially the distance from the point to the x-axis. So, all the points on the x-axis have a y-coordinate of 0.

For example, in the graph below, the x-intercept is the point $(0,2)$ and the y-intercept is the point $(-6,0)$.



Usually, the points on the coordinate axes are labeled with only their non-zero coordinate. We'll follow this convention in the next set of examples.

In previous examples, we often substituted 0 in the linear equations to find our first point to plot. Each time we did this, we found one of the intercepts. Now, we will plot each line by finding both its x-intercept and its y-intercept.

Extended Example 3a

Graph this line by finding its x and y-intercepts: $3y+5x=-30$.

EXAMPLE B

Find the y-intercept of the line $\frac{2x-3}{5}=\frac{7y-4}{2}$.

To find the y-intercept, set $x=0$, and solve for y:

$$\frac{2\cdot x-3}{5}=\frac{7y-4}{2}$$
$$\frac{2\cdot 0-3}{5}=\frac{7y-4}{2}$$
$$\frac{-3}{5}=\frac{7y-4}{2}$$

Multiply both sides by 10, the least common multiple of the denominators:

$$10\cdot\frac{-3}{5}=10\cdot\frac{7y-4}{2}$$
$$\frac{10\cdot(-3)}{5}=\frac{10\cdot(7y-4)}{2}$$
$$\frac{2\cdot\cancel{5}\cdot(-3)}{\cancel{5}}=\frac{\cancel{2}\cdot 5\cdot(7y-4)}{\cancel{2}}$$
$$-6=35y-20$$

(***Note:*** Cross-multiplying will give you the same result.)

continued...

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Example B, continued…

Now solve $-6 = 35y - 20$ for y:

$$
\begin{aligned}
-6 &= 35y - 20 \\
+20 &\qquad\;\; +20 \\
\hline
14 &= 35y \\
\frac{14}{35} &= \frac{35y}{35} \\
\frac{2 \cdot \cancel{7}}{5 \cdot \cancel{7}} &= y \\
\frac{2}{5} &= y
\end{aligned}
$$

The y-intercept is the point $\left(0, \frac{2}{5}\right)$.

Extended Example 4a

Find the y-intercept of the line $\frac{4y-1}{3}=\frac{3x+2}{9}$.

END OF LESSON

Graph: $2x + 3y = 0$

Graph the following line.
$3x - 4y = 0$

Find the slope: $2x + 7y = 4$

Graph the following line.

$2y - 3x = -2$

Are these lines parallel, perpendicular, or neither: $y = 4x + 2$ and $2x - 8y = 0$?

SLOPE AND THE SLOPE-INTERCEPT FORM

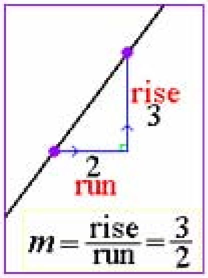
Introduction

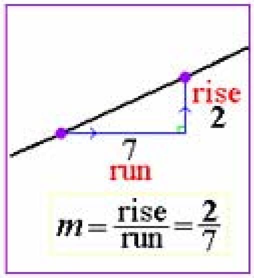
This lesson explains the concept of **slope** and introduces the single most useful form for linear equations – the **slope-intercept form**. In the slope-intercept form, the algebra and geometry come together nicely.

SLOPE AND THE SLOPE-INTERCEPT FORM

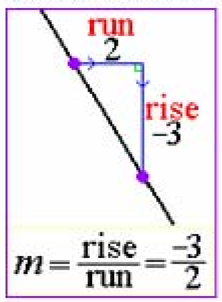
Slope of a Line

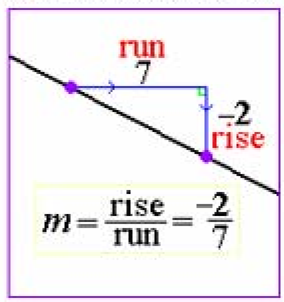
We'll start with the geometric notion of the slope of a line. The **slope of a line** is the "rise" divided by the "run." In the following examples, we start at a point on the line and "run" off the line horizontally to the right. Then we "rise" to get back on the line. The slope, m , becomes the rise divided by the run. (Traditionally, m is used for slope; it is thought to derive from the French word "monter," which means "to climb.")





If the rise is actually a fall (going down instead of up), then the rise has a negative sign. If the line has a positive run, the slope has the same sign as the rise. If the rise is positive, then so is the slope, and if the rise is negative, the slope is, too. In this course, we'll always assume that whenever the slope is negative we have a negative rise divided by a positive run.





SLOPE AND THE SLOPE-INTERCEPT FORM

Horizontal and Vertical Lines

The slope of a horizontal line is 0 (as you might expect), since the rise is 0 and

$$\frac{0}{\text{any number}} = 0.$$

The slope of a vertical line is not defined, since the run is always 0 and

$$\frac{\text{any number}}{0} = \text{undefined}.$$

Slope Formula

To find the slope of the line through two given points, (x_1, y_1) and (x_2, y_2), you can use the slope formula:

$$\text{Slope} = m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{rise}}{\text{run}}$$

The difference of the y-coordinates is the rise (notice that "rise" rhymes with "y's"), and the difference of the x-coordinates is the run.

EXAMPLE A

Find the slope of the line through the points $(2,3)$ and $(6,5)$.

The run between these points is the difference of their x-coordinates,

$$\text{run} = 6 - 2 = 4,$$

and the rise between these points is the difference of their y-coordinates,

$$\text{rise} = 5 - 3 = 2.$$

So, the slope is $\frac{\text{rise}}{\text{run}} = \frac{2}{4} = \frac{1}{2}$.

Or, we can use the slope formula with our two points:

$$\begin{matrix} (x_1, & y_1) & & (x_2, & y_2) \\ \| & \| & \& & \| & \| \\ (2, & 3) & & (6, & 5) \end{matrix}$$

Substituting these values, we get:

$$\text{Slope} = m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-3}{6-2} = \frac{2}{4} = \frac{\cancel{2}}{\cancel{2} \cdot 2} = \frac{1}{2}$$

Note:

- There's no preferred order for the points in calculating the slope. Our next example presents the same two points in the opposite order but results in the same slope that we found here.

EXAMPLE B

Find the slope of the line through the points $(6,5)$ and $(2,3)$.

These are the same two points as were given in Example A, but in a different order. Using the slope formula we have:

$$\begin{matrix} (x_1, y_1) & & (x_2, y_2) \\ \| \quad \| & \& & \| \quad \| \\ (6, \quad 5) & & (2, \quad 3) \end{matrix}$$

$$\text{Slope} = m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-5}{2-6} = \frac{-2}{-4} = \frac{\cancel{-2}}{\cancel{-2} \cdot 2} = \frac{1}{2}.$$

Question: Find the slope of the line through the points $(-4,2)$ and $(3,-7)$.

Note:

- It's helpful to put negative numbers in parentheses when using the slope formula, to keep the signs clear.
- In this course, we'll always give the negative sign to the rise when the slope is negative.

Question: Find the slope of the line through the points $(-12,-19)$ and $(14,21)$.

Question: Find the slope of the line through the points $(-2.1,5.1)$ and $(9.4,-1.8)$.

SLOPE AND THE SLOPE-INTERCEPT FORM

EXAMPLE C

Find the slope of the line $y = 3x - 5$.

Find two points on this line; then, calculate the slope of the line through those points. We can find the y-intercept by setting $x = 0$. Then we'll set $x = 1$ and find its corresponding y-coordinate.

$y = 3x - 5$	x	y	Point
$y = 3 \cdot 0 - 5$ $y = -5$	0	-5	$(0, -5)$
$y = 3 \cdot 1 - 5$ $y = 3 - 5$ $y = -2$	1	-2	$(1, -2)$

Instead of graphing this line, use the slope formula to calculate its slope:

$$\text{Slope} = m = \frac{(-2) - (-5)}{1 - 0} = \frac{-2 + 5}{1} = 3$$

Note:

- The line $y = 3x - 5$ crosses the y-axis at $(0, -5)$, and its slope is 3. When the equation of a line is written in a form like $y = 3x - 5$, the slope and the y-intercept can both be seen directly in the equation:

slope
↓
$$y = 3x - 5$$
↑
y–intercept

SLOPE AND THE SLOPE-INTERCEPT FORM

Slope-Intercept Form

When a line is written in the form $y = mx + b$, then the slope of the line is m and the y-intercept is $(0, b)$. For this reason, it's called the **slope-intercept form** for the equation of a line.

slope ↓

$$y = mx + b$$

↑ y–intercept

When the line is written in this form, the geometry is evident in the algebra.

As was mentioned earlier, a vertical line has an undefined slope; it is the only type of line with an equation that cannot be written in slope-intercept form.

EXAMPLE D

Find the slope and y-intercept of the line $y = -4x + 2$.

Since $y = -4x + 2$ is in slope-intercept form, it has a slope of -4 and its y-intercept is $(0, 2)$.

When the linear equation is in slope-intercept form, the slope and y-intercept are right in front of your nose!

SLOPE AND THE SLOPE-INTERCEPT FORM

EXAMPLE E

Find the slope and y-intercept of the line $6x-3y=18$.

This time the equation isn't in slope-intercept form. To get it into this ideal form, solve the equation for y.

First, isolate the y term by subtracting $6x$ from both sides:

$$\begin{array}{rcl} 6x-3y & = & 18 \\ -6x & & -6x \\ \hline -3y & = & -6x+18 \end{array}$$

Now solve for y by dividing both sides by -3:

$$\frac{-3y}{-3}=\frac{-6x+18}{-3}$$

$$\frac{\cancel{-3}y}{\cancel{-3}}=\frac{-6x}{-3}+\frac{18}{-3}$$

$$y=2x-6$$

Now the equation $y=2x-6$ is in slope-intercept form. The slope is 2 and the y-intercept is $(0,-6)$.

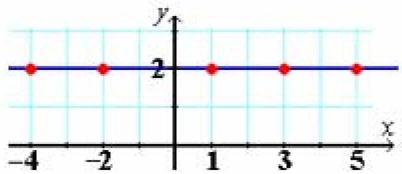
SLOPE AND THE SLOPE-INTERCEPT FORM

Extended Example 1a

Find the slope and y-intercept of the line $7x - 2y = 14$.

Horizontal Lines

Again, the slope of a horizontal line is zero.



$y = 2$

Notice that all the points on this line have a y-coordinate of 2. If a point has a y-coordinate of two, it lies on this line. The x-coordinate can be any number. So, it makes sense to call the equation for this line $y = 2$. Finding the slope-intercept form for this line confirms this equation. First we compute the slope using two points on the line—for example, let's use $(1,2)$ and $(3,2)$:

$$\text{Slope} = m = \frac{2-2}{3-1} = \frac{0}{2} = 0$$

The slope is 0. It's clear from the graph that the y-intercept is the point $(0,2)$. Therefore, the slope-intercept equation of this line is:

$$y = m \cdot x + 2$$
$$y = 0 \cdot x + 2$$
$$y = 0 + 2$$
$$y = 2$$

The equation for any horizontal line can be written in the form $y = C$, where C is some constant.

SLOPE AND THE SLOPE-INTERCEPT FORM

EXAMPLE F

Find the equation of the horizontal line through the point $(3,-1)$.

All horizontal lines are of the form $y = C$, for some constant C. This horizontal line goes through the point $(3,-1)$. Since the y-coordinate is -1, the equation of this line must also be $y = -1$.

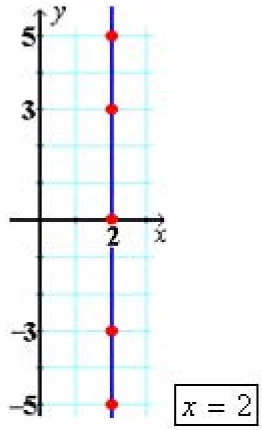
Question: Find the equation of the horizontal line through the point $(-4,3)$.

Question: Find the equation of the horizontal line through the point $(5,7)$.

SLOPE AND THE SLOPE-INTERCEPT FORM

Vertical Lines

Vertical lines have undefined slopes. For example, consider the vertical line shown below.



All the points on this line have an x-coordinate equal to 2. If the x-coordinate of a point is two, it lies on this line. So, it makes sense to call the equation for this line $x = 2$. Now let's see what we get when we compute the slope using the two points $(2,-3)$ and $(2,3)$:

$$\text{Slope} = m = \frac{3-(-3)}{2-2} = \frac{6}{0} = \text{undefined}$$

The slope is undefined. We find that the equation of a vertical line cannot be put into slope-intercept form ($y = mx + b$).

Any vertical line has an equation that can be written in the form $x = C$, where C is some constant.

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SLOPE AND THE SLOPE-INTERCEPT FORM

EXAMPLE G

Find the equation of the vertical line through the point $(3,-1)$.

All vertical lines are of the form $x = C$, for some constant C. This vertical line goes through the point $(3,-1)$, and the x-coordinate is 3. So, the equation of this line must be $x = 3$.

Question: Find the equation of the vertical line through the point $(9,2)$.

Question: Find the equation of the vertical line through the point $(-4,3)$.

Question: Find the equation of the vertical line through the point $(5,7)$.

EXAMPLE H

Find the equation of the line with slope 8, through the point $(0,-3)$.

The given point is, in fact, the y-intercept (since its x-coordinate is 0). In slope-intercept form, this line's equation is $y = 8x - 3$.

EXAMPLE I

Find the equation of the line with slope 2 that passes through the point $(1,5)$.

We know the slope is 2. So the equation must be of the form $y = 2x + b$, for some b. To find the y-intercept, b, use the given point $(1,5)$ with $x = 1$ and $y = 5$. If we substitute these values into the equation, we can solve for b:

$$x = 1,\ y = 5 \quad \rightarrow \quad \begin{aligned} y &= 2x + b \\ 5 &= 2 \cdot 1 + b \\ 5 &= 2 + b \\ 3 &= b \end{aligned}$$

The y-intercept is 3, so the equation of this line is $y = 2x + 3$.

SLOPE AND THE SLOPE-INTERCEPT FORM

Extended Example 2a

Find the equation of the line with slope 10 that passes through the point $(2,3)$.

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SLOPE AND THE SLOPE-INTERCEPT FORM

EXAMPLE J

Find the equation of the line through the points $(-4,2)$ and $(3,-1)$.

We are not given the slope this time. Use the two given points to find the slope:

$$\text{Slope} = m = \frac{(-1)-2}{3-(-4)} = \frac{-3}{3+4} = -\frac{3}{7}.$$

Now that we have the slope, we know our line has the equation $y = -\frac{3}{7}x + b$.

To find b, substitute either of the given points into this equation. The point $(3,-1)$ has an x-coordinate of 3 and a y-coordinate of -1.

$$x = 3,\ y = -1 \quad \rightarrow \quad y = -\frac{3}{7}\cdot x + b$$

$$-1 = -\frac{3}{7}\cdot 3 + b$$

$$-1 = -\frac{9}{7} + b$$

$$-1 + \frac{9}{7} = b$$

$$-\frac{7}{7} + \frac{9}{7} = b$$

$$\frac{2}{7} = b$$

The equation of this line is $y = -\frac{3}{7}x + \frac{2}{7}$.

SLOPE AND THE SLOPE-INTERCEPT FORM

Extended Example 3a

Find the equation of the line through the points $(-2,-5)$ and $(2,3)$.

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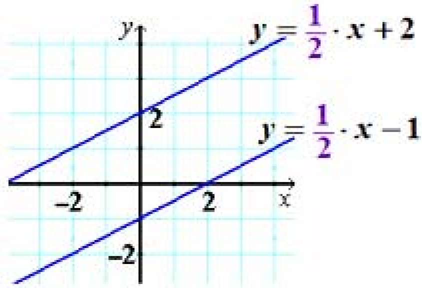
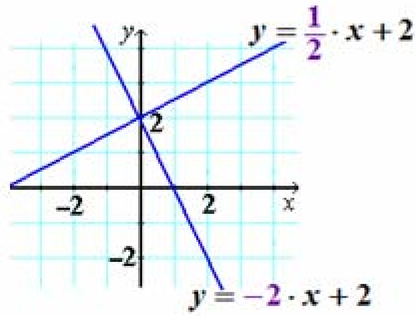
SLOPE AND THE SLOPE-INTERCEPT FORM

Parallel Lines

Lines with the same slope are **parallel**. For example, $y = 2x$, $y = 2x - 3$, and $y = 2x + 7$ are parallel lines since they all have a slope equal to 2. The symbol $||$ is often used for "is parallel to."



parallel lines



perpendicular lines

Perpendicular Lines

Two lines are **perpendicular** if the product of their slopes is -1. The slopes of such lines are negative reciprocals of each other. The symbol $\perp$ is often used for "is perpendicular to." To find a slope perpendicular to a given slope, take its reciprocal and change its sign. For example:

$$y = 2x - 3 \perp y = -\frac{1}{2}x + 7 \quad \text{because} \quad 2 \cdot \left(-\frac{1}{2}\right) = -1$$

$$y = -\frac{3}{5}x + 2 \perp y = \frac{5}{3}x - 3 \quad \text{because} \quad \left(-\frac{3}{5}\right) \cdot \left(\frac{5}{3}\right) = -1$$

$$y = \frac{1}{8}x - 6 \perp y = -8x + 1 \quad \text{because} \quad \left(\frac{1}{8}\right) \cdot (-8) = -1$$

SLOPE AND THE SLOPE-INTERCEPT FORM

EXAMPLE K

Determine whether the following lines are parallel, perpendicular, or neither.

$$2x-3y=24 \text{ and } y=-\frac{3}{2}x+1.$$

The second equation is in slope-intercept form, and we can see that its slope is $m=-\frac{3}{2}$. To find the slope of the first line, put the first equation into slope-intercept form by solving for y:

$$\begin{aligned} 2x-3y &= 24 \\ -2x \quad & \quad -2x \\ \hline -3y &= -2x+24 \\ \frac{-3y}{-3} &= \frac{-2x+24}{-3} \\ y &= \frac{-2x}{-3}+\frac{24}{-3} \\ y &= \frac{2}{3}x-8 \end{aligned}$$

The first line has slope $\frac{2}{3}$, while the second line has slope $-\frac{3}{2}$. Since these slopes are negative reciprocals of each other (the product of their slopes is -1), these lines are perpendicular.

Extended Example 4a

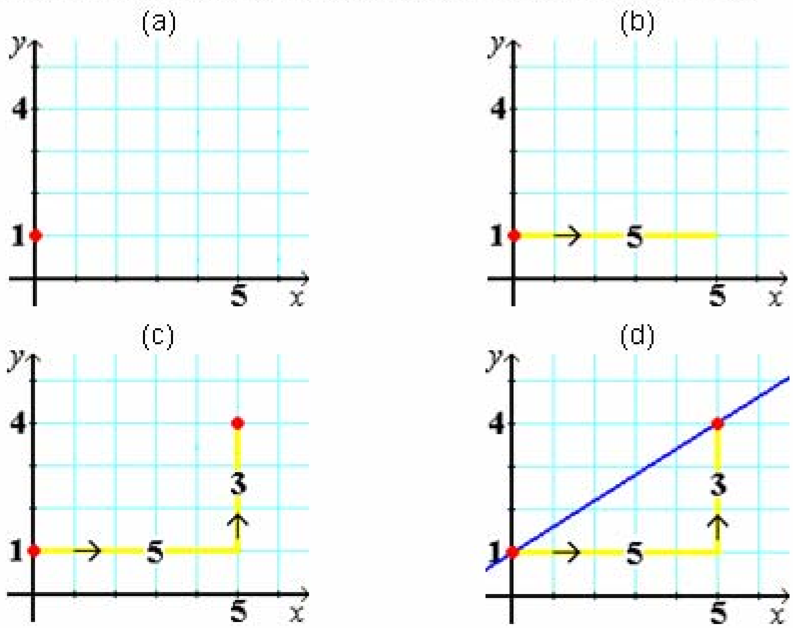
Determine whether the lines $8x+10y=1$ and $y=-\frac{4}{5}x+2$ are parallel, perpendicular, or neither.

SLOPE AND THE SLOPE-INTERCEPT FORM

Graphing Lines Directly from Slope-Intercept Form

It's often quite easy to graph a line of an equation in slope-intercept form without making the usual table of x, y values. For example, let's graph the line

$y = \frac{3}{5}x + 1$. Right away we know that the line has y-intercept $(0,1)$, so we can start by plotting this point [see (a), below]. The line has slope $m = \frac{3}{5} = \frac{\text{rise}}{\text{run}}$. If we "run" a distance of 5 from the y-intercept [see (b), below], we must "rise" a distance of 3 [see (c), below] to get back to the line [see (d), below]:



SLOPE AND THE SLOPE-INTERCEPT FORM

Extended Example 5a

Graph the line $y = \frac{4}{3}x + 2$ using only this line's slope and y-intercept.

END OF LESSON

Graph, and state the slope and y-intercept: $5 = 3x - y$

Find the slope and y- intercept of the following line.

$5 = 3x - y$

Find the slope and $y-$intercept: $2x - 3y = 0$

Write the equation of the line with slope $-\frac{3}{4}$ and y-intercept 3

Find the x- and y-intercepts: $2x - y = 8$

State whether the given lines are parallel, perpendicular, or neither.

$8x - 5y = 40$ and $5x - 8y = 40$

POINT-SLOPE FORM

Introduction

In this section we'll examine another form of linear equations—the **point-slope form**. Although you can solve any problem involving linear equations of non-vertical lines using the slope-intercept form learned earlier, the point-slope form will help to deepen your understanding of the connections between algebra and geometry.

POINT-SLOPE FORM

Let's start with an example using the slope-intercept formula, $y = mx + b$, where m is the slope and b is the y-intercept.

EXAMPLE A

Find the line with slope 3 that passes through the point $(1,2)$.

Start with the slope-intercept formula, $y = mx + b$, and substitute the known slope, $m = 3$:

$$y = 3x + b.$$

Then, substitute in the x and y-coordinates of the given point, $(1,2)$:

$$x = 1,\ y = 2 \quad \rightarrow \quad \begin{aligned} y &= 3 \cdot x + b \\ 2 &= 3 \cdot 1 + b \\ 2 &= 3 + b \\ -1 &= b \end{aligned}$$

So, our line is $y = 3x - 1$.

Next, we'll redo this example in a way that will lead to the point-slope form for the equation of this line.

POINT-SLOPE FORM

EXAMPLE B

Find the line with slope 3 that passes through the point $(1,2)$.

This time, suppose that (x,y) is another point on this line. We can now compute the slope between the two points on our line, $(1,2)$ and (x,y):

$$m = \frac{y-2}{x-1}.$$

Note that we are told that this slope equals 3:

$$\frac{y-2}{x-1} = 3.$$

Multiply both sides of this equation by $(x-1)$ to clear the denominator:

$$\frac{y-2}{x-1} \cdot (x-1) = 3 \cdot (x-1)$$

$$\frac{(y-2) \cdot \cancel{(x-1)}}{\cancel{(x-1)}} = 3 \cdot (x-1)$$

$$y-2 = 3(x-1)$$

That last line is the equation of the given line in point-slope form.

The general formula for the **point-slope form** of a line with slope m that passes through point (x_1, y_1) is:

$$y - y_1 = m(x - x_1).$$

POINT-SLOPE FORM

EXAMPLE C

Find the point-slope form of the line with slope $2/3$ that passes through the point $(3,5)$.

We are given $m = 2/3$ and $(x_1, y_1) = (3,5)$.

Substitute these values into the point-slope formula:

$$m = \frac{2}{3}, \quad x_1 = 3, \quad y_1 = 5$$

$$y - y_1 = m(x - x_1)$$

$$y - 5 = \frac{2}{3}(x - 3)$$

The equation in point-slope form is $y - 5 = \frac{2}{3}(x - 3)$.

POINT-SLOPE FORM

Question: Find the point-slope form of the line with slope $5/2$ that passes through the point $(-1,9)$.

Question: Find the point-slope form of the line with slope -4 that passes through the point $(-8,-3)$.

POINT-SLOPE FORM

Question: Find the point-slope form of the line with slope $-\frac{7}{11}$ that passes through the point $\left(\frac{2}{3},-\frac{3}{2}\right)$.

POINT-SLOPE FORM

EXAMPLE D

Find the equation of the line parallel to $y = 3x - 66$ that passes through the point $(4, -2)$. Express your answer in point-slope form.

Parallel lines have the same slope. So, we know that the line we need to find has $m = 3$, and we know that $(x_1, y_1) = (4, -2)$.

Substitute the values into the point-slope formula:

$$m = 3, \quad x_1 = 4, \quad y_1 = -2$$

$$y - y_1 = m(x - x_1)$$

$$y - (-2) = 3(x - 4)$$

$$y + 2 = 3(x - 4)$$

POINT-SLOPE FORM

Extended Example 1a

Find the line parallel to $y - 3x = 5$ that passes through the point $(1,1)$.
Express your answer in point-slope form.

POINT-SLOPE FORM

EXAMPLE E

Find the equation of the line perpendicular to $y = \frac{1}{5}x - 33$ that passes through the point $(3, -4)$. Express your answer in point-slope form.

Perpendicular lines have negative reciprocal slopes.

The negative reciprocal of $\frac{1}{5}$ is $-\frac{5}{1}$, or -5, so our line has slope $m = -5$ and $(x_1, y_1) = (3, -4)$.

Substitute these values into the point-slope formula:

$$m = -5, \quad x_1 = 3, \quad y_1 = -4$$
$$y - y_1 = m(x - x_1)$$
$$y - (-4) = -5(x - 3)$$
$$y + 4 = -5(x - 3)$$

POINT-SLOPE FORM

Extended Example 2a

Find the line perpendicular to $y - 6x = 1$ that passes through the point $(-1, -4)$. Express your answer in point-slope form.

POINT-SLOPE FORM

EXAMPLE F

Find the point-slope equation of the line that passes through the points $(2,3)$ and $(-1,-3)$.

First, we find the slope of the line through the two given points:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-3) - 3}{(-1) - 2} = \frac{-6}{-3} = 2$$

Next, let's pick the first of the two points given, $(2,3)$, and substitute it and the slope $m = 2$ into the point-slope formula:

$$m = 2, \quad x_1 = 2, \quad y_1 = 3$$
$$y - y_1 = m(x - x_1)$$
$$y - 3 = 2(x - 2)$$
$$y - 3 = 2(x - 2)$$

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POINT-SLOPE FORM

Extended Example 3a

Find the point-slope equation of the line through the points $(-3, 6)$ and $(-2, -9)$.

POINT-SLOPE FORM

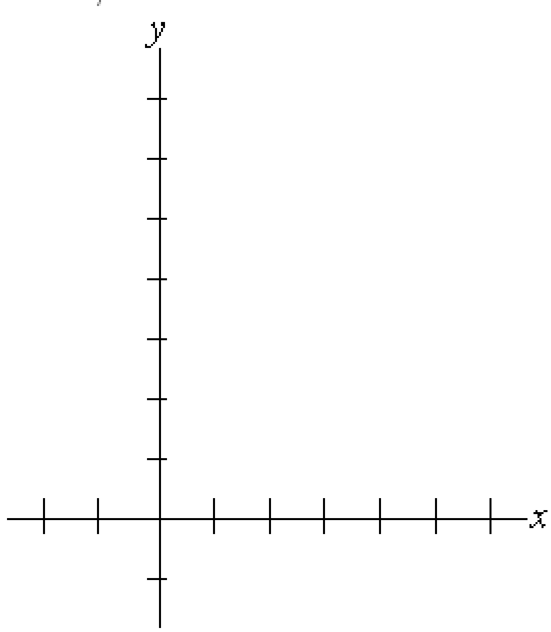
Extended Example 4a

Use the point-slope formula to find an equation of the line that passes through the point $(3,2)$ and has a slope of $-\frac{4}{3}$. Then put this equation into slope-intercept form.

END OF LESSON

Plot the points, draw the graph, determine the equation, and give the slope and the y-intercept: $(-2,3)$, $(1,-2)$

Plot the point, draw the graph, determine the equation in slope-intercept form, and give the y-intercept: Through $(5,5)$ with slope $m = \frac{2}{7}$.



Find the point-slope equation of the line through point $(-4,12)$ and parallel to the line $2x - 7y = 13$.

Plot the point, draw the graph, determine the equation in slope-intercept form, and give the y-intercept: Through $(5,5)$ with slope $m = \frac{2}{7}$.

Find the point-slope equation of the line through point $(2,4)$ and perpendicular to the line $y = \frac{9}{7}x - \frac{7}{2}$.

Find a point-slope equation of the line through points $(13,10)$ & $(19,-11)$. (Hint: Use the first point!)

RELATIONS AND FUNCTIONS

Introduction

In this lesson, we'll study **relations** and **functions**. The concept of a "function" underlies much of modern science and technology, and is the basis for advanced work in mathematics. Not everything in the world can be modeled using lines, but the graphs of relations and functions fit most situations.

RELATIONS AND FUNCTIONS

Sets

A **set** is a collection of things called **elements** that are enclosed in curly brackets $\{\ \}$ and separated by commas. For example, $\{a,b,c\}$ is a set that contains the elements $a,b,$ and c. The listing order of the elements does not matter: $\{a,b,c\}=\{c,a,b\}$.

Repetition of elements is usually avoided, although the set $\{a,a,b,c\}$ is considered the same set as $\{a,b,c\}$. A set that allows repetition of one or more of its elements is called a **multiset**.

Note:

- In this course, we will avoid the use of repetitious elements when listing a set.

Ordered Pairs

An **ordered pair** consists of two items separated by a comma and enclosed in parentheses. Unlike sets, order <u>does</u> matter here: $(1,2)\neq(2,1)$. When both elements of on ordered pair are numbers, think of the ordered pair as the x,y coordinates of a point. In this lesson, ordered pairs will usually be points on a plane.

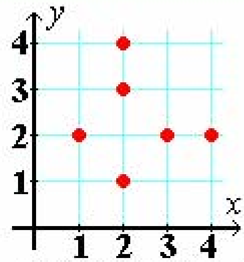
Relations

A **relation** is a set of ordered pairs. A relation doesn't necessarily involve numbers. Marriage can be considered a relation (let the first element of each ordered pair be a wife, and let the second element be her husband). In this lesson, relations can be thought of as sets of points. To graph a relation, just plot its points.

EXAMPLE A

Graph the relation $R = \{(1,2),(2,1),(2,3),(3,2),(2,4),(4,2)\}$.

Plot each of the points in R:



Graph of relation R

Domain and Range

The **domain** of a relation is the set of all first coordinates (or x-coordinates) in the relation.

The **range** of a relation is the set of all second coordinates (or y-coordinates) in the relation.

For example, let $S = \{(1,2),(3,4),(5,6),(1,6),(3,2)\}$ be a relation.

Then "the domain of $S = \{1,3,5\}$ and "the range of $S = \{2,4,6\}$.

Question: Find the domain and range of the relation:

$\{(\text{red, Mars}), (\text{violet, Saturn}), (\text{green, Venus}), (\text{yellow, Mercury})\}$

Question: Find the domain and range of the relation:

$\{(\text{cuddly, kitten}), (\text{cuddly, puppy}), (\text{prickly, porcupine})\}$

Question: Find the domain and range of the relation:

$\{(a,b), (a,c), (b,b), (b,c), (b,d)\}$

RELATIONS AND FUNCTIONS

Functions

A **function** is a relation in which each element of the domain occurs only once as a first coordinate of the relation. A function has no repetitions of any of its x-coordinates. Study these examples:

$\{(a,b),(b,c),(c,b)\}$ is a function, since each domain element occurs once as a first coordinate.

$\left\{\begin{matrix}\downarrow & \downarrow & \\ (a,b), & (a,c), & (c,b)\end{matrix}\right\}$ is not a function, since a occurs twice as a first coordinate.

$\{(0,2),(1,2),(2,2)\}$ is a function, since each domain element occurs once as a first coordinate.

$\left\{\begin{matrix}\downarrow & & \downarrow \\ (1,1), & (2,2), & (1,2)\end{matrix}\right\}$ is not a function, since 1 occurs twice as a first coordinate.

Notice that we don't care about the second coordinate at all in deciding whether a relation is a function.

RELATIONS AND FUNCTIONS

Question: Is the relation $\{(0,1),(1,1),(2,1),(3,2),(4,2)\}$ a function?

Question: Is the relation $\{(0,0),(0,1),(1,3),(2,5)\}$ a function?

Question: Is the relation $\{(1,4),(4,1),(2,1),(1,2)\}$ a function?

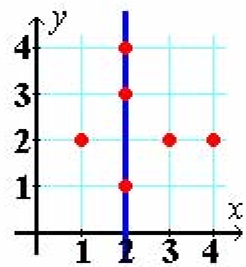
Vertical Line Test

There's an easy way to tell if a relation is a function by looking at its graph. A vertical line can never cross the graph of a function more than once. If any vertical line crosses the graph of a relation more than once, then that relation is not a function.

Consider the relation we encountered earlier, $R = \{(1,2),(2,1),(2,3),(3,2),(2,4),(4,2)\}$. This relation is not a function, since the element 2 is repeated more than once as a first coordinate:

$$\left\{(1,2),(\overset{\downarrow}{2},1),(\overset{\downarrow}{2},3),(3,2),(\overset{\downarrow}{2},4),(4,2)\right\}.$$

This repetition can be seen in the graph of R. The vertical line depicted crosses R three times, one for each time 2 was repeated as a first coordinate.



The first coordinate is the x-coordinate. If two points have the same x-coordinate, they are on the same vertical line. Conversely, if two points are on the same vertical line, they share the same x-coordinate, and cannot be on the graph of a function.

Input-Output and Function Notation

The domain of a function can be thought of as the set of all its possible **inputs** with the range of the function as its corresponding **outputs**. One input yields only one output. The input is the x-coordinate, and this input is paired with its corresponding output, the y-coordinate.

Consider the function $f=\{(1,5),(2,4),(3,3),(4,5)\}$. The function f can be visualized as an input-output table:

f

Input		Output
1	$\longrightarrow$	5
2	$\longrightarrow$	4
3	$\longrightarrow$	3
4	$\longrightarrow$	5

Functions have a special notation. Suppose the point (x,y) is on the graph of a function f. Inputting x results in output y. In function notation, this is expressed as

$$f(x)=y.$$

(This is read as: "f of x equals y")

RELATIONS AND FUNCTIONS

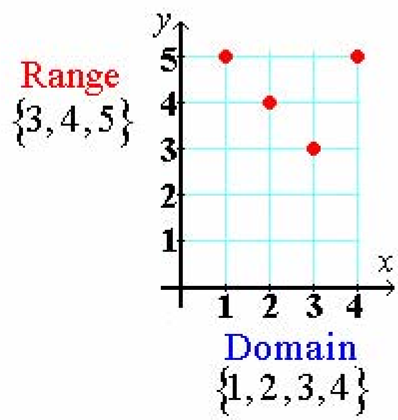
The input-output table, including function notation, for the function $f = \{(1,5),(2,4),(3,3),(4,5)\}$ from the previous screen would look like this:

Function f

Input x		**Output** y	**Function Notation** $f(x) = y$
1	$\longrightarrow$	5	$f(1) = 5$
2	$\longrightarrow$	4	$f(2) = 4$
3	$\longrightarrow$	3	$f(3) = 3$
4	$\longrightarrow$	5	$f(4) = 5$

Graph of the function

$f = \{(1,5),(2,4),(3,3),(4,5)\}$:



WARNING: The notation $f(x)$ is <u>only</u> used to represent functions. It does <u>not</u> mean multiply f times x. Always read $f(x)$ as "f of x."

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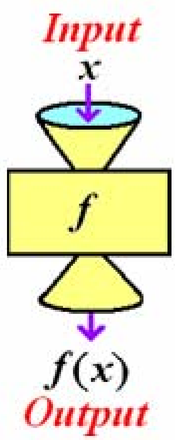
RELATIONS AND FUNCTIONS

When thinking of functions, it may be helpful to imagine the image of the "function box" below.

"Domain of f"		**"Range of f"**
= "Set of all possible <u>inputs</u> to f" = "Set of all x-coordinates of points on the graph of f" The variable x is sometimes referred to as the **independent variable**.		= "Set of all possible <u>outputs</u> of f" = "Set of all y-coordinates of points on the graph of f" The variable $y = f(x)$ is sometimes referred to as the **dependent variable**. It is called "dependent" because the output y depends on the input x.

Usually, the variables f, g, and h (or F, G, and H), are used for functions. However, any variable can represent a function.

RELATIONS AND FUNCTIONS

Function notation is particularly useful to define functions with infinite domains. For example, consider the function defined by the equation:

$$y = g(x) = 2x - 1 .$$

This is the equation of a line in the slope-intercept form, $y = 2x - 1$. You can think of y as a function of x . This tells you what g does to an input x to obtain its corresponding output y . This function has all real numbers for its domain, since any real number may be substituted for x .

In function notation, input 5 by substituting it for x :

$$\begin{aligned} y &= g(x) = 2x - 1 \\ y &= g(5) = 2 \cdot 5 - 1 \\ &= 10 - 1 \\ y &= g(5) = 9 \end{aligned}$$

It's helpful to think of a function as a process. In this case, our function g is the process $2x - 1$, which means "double the input, and then subtract 1 ." If 5 is the input to this function, it first gets doubled, becoming 10 . Then 1 is subtracted, yielding an output of 9 .

So, the point $(5, 9)$ is on the graph of g . We can input any real number, and find the resulting output. This process is called **evaluating the function**.

It's interesting to note that function notation resembles the function box we introduced on the previous screen:

input

$$f(\underset{\uparrow}{x}) = \xrightarrow{\text{output}} y$$

RELATIONS AND FUNCTIONS

EXAMPLE B

Evaluate function $g(x) = 2x - 1$ at inputs 0, $\frac{1}{2}$, and -3. What are the corresponding points on the graph of g?

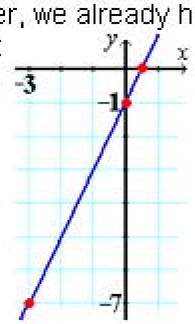
For each of these, x is the input and y is the output .

$$g(0) = 2 \cdot 0 - 1$$
$$= 0 - 1 \longrightarrow \text{The point } (0, -1) \text{ is on the graph of } g.$$
$$= -1$$

$$g(\tfrac{1}{2}) = 2 \cdot (\tfrac{1}{2}) - 1$$
$$= 1 - 1 \longrightarrow \text{The point } (\tfrac{1}{2}, 0) \text{ is on the graph of } g.$$
$$= 0$$

$$g(-3) = 2 \cdot (-3) - 1$$
$$= -6 - 1 \longrightarrow \text{The point } (-3, -7) \text{ is on the graph of } g.$$
$$= -7$$

By defining the function g as $g(x) = 2x - 1$, we have a way to find many more points on this function's graph. However, we already have enough points to graph the infinite line since g is linear:



$g(x) = 2x - 1$

Extended Example 1a

Evaluate the function $h(x) = 3x - 4$ at $x = 2$. What is the corresponding point on the graph of h?

Functions that have lines as graphs are called **linear functions**. Any equation of a line written in slope-intercept form can be thought of as a linear function. In that form, y is written as a function of x. Our next example involves a non-linear function.

EXAMPLE C

Evaluate the function $F(x) = \dfrac{x^2 - 2}{2x}$ at $x = 5$. Which point on the graph of F have we found?

$$F(x) = \frac{x^2 - 2}{2 \cdot x}$$

$$F(5) = \frac{5^2 - 2}{2 \cdot 5}$$

$$= \frac{25 - 2}{10}$$

$$F(5) = \frac{23}{10}$$

The point $\left(5, \dfrac{23}{10}\right)$ is on the graph of F.

Note:

- If we plot many points of a non-linear function, we can graph the function by connecting the dots. This can be a tedious process. Luckily, computers and graphing calculators can perform this task very easily!

Extended Example 2a

Evaluate the function $f(x) = x^2 - 5$ at $x = 4$. What point on the graph of f have we found?

RELATIONS AND FUNCTIONS

EXAMPLE D

For the function $f(x) = 5 - 9x$, what input will yield an output of 100? In other words, if $f(x) = 100$ then what is x?

We know that $f(x) = 5 - 9x$.

So:
$$f(x) = 100$$
$$5 - 9x = 100$$

Now we solve $5 - 9x = 100$ for x:

$$\begin{aligned} 5 - 9x &= 100 \\ -5 \qquad & \;\; -5 \\ \hline -9x &= 95 \\ \frac{-9x}{-9} &= \frac{95}{-9} \\ x &= -\frac{95}{9} \end{aligned}$$

So, for the function $f(x) = 5 - 9x$, the input $-\frac{95}{9}$ will give an output of 100.

We can check this:

$$5 - 9\left(-\frac{95}{9}\right) = 5 - \cancel{9}\left(-\frac{95}{\cancel{9}}\right) = 5 - (-95) = 5 + 95 = 100.$$

Question: For the function $g(x) = 10x - 13$, what input will yield an output of 22?

Question: For the function $H(x) = 17 - 23x$, what input yields an output of 0?

RELATIONS AND FUNCTIONS

Question: For the function $f(x) = 321 + 123x$, what input yields an output of 111?

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Finding a Function's Domain

How do you determine the domain of a function?

For the linear functions we've been looking at, the domain is the entire set of real numbers.

Sometimes the line comes from a real-world problem. For example, y might represent the number of buffalo on a prairie. Although the line may have points where y is negative, you can't have a negative number of buffalo! In such a case you would, reasonably, decide to exclude values of x that result in such negative y values. Use common sense.

When you define a function, you may state the function's domain as long as the function is well defined in the specified domain. A function depends on its "rule" and its domain. If a domain isn't specified for a function, assume the domain is as large as possible and then work from there.

When you are asked to find the domain of a function, it helps to ask yourself first, "What's not in the domain of the function?" Those exceptions are usually easy to spot.

Be on the lookout for the following values to exclude from a function's domain:

- Any values that make a function's denominator zero (undefined)
- Values of x that result in the square root of a negative number
- When applying functions to the real world, any input values that result in meaningless or unrealistic outputs (if graphing a number of real things, you can't have a negative number)

EXAMPLE E

Find the domain of the function $f(x) = \dfrac{1}{x+3}$.

Since division by 0 is undefined, this function is undefined when $x + 3 = 0$.

$$\begin{aligned} x + 3 &= 0 \\ -3 \quad &\;\; -3 \\ \hline x &= -3 \end{aligned}$$

The domain of $f(x) = \dfrac{1}{x+3}$ is "all real numbers, <u>except</u> -3."

This can be expressed in the following so-called **set-builder notation**:

$$\left\{ x \,\middle|\, x \neq -3 \right\}$$

The vertical line that follows the first x means "such that." To the right of that vertical line go any conditions that x must satisfy to be a member of that set.

So, $\left\{ x \,\middle|\, x \neq -3 \right\}$ is read "the set of all real numbers x such that x doesn't equal negative three."

Extended Example 3a

Find the domain of the function $H(x) = \dfrac{5x-2}{5+2x}$.

EXAMPLE F

Find the domain of the function $F(x) = \sqrt{7-8x}$.

To avoid a negative value under the square root, the radicand, $7-8x$, must be positive or its square root won't be a real number. In other words, it must be greater than or equal to zero. We must solve for x when $7-8x$ is greater than or equal to 0:

$$7 - 8x \geq 0$$
$$-7 \qquad -7$$
$$-8x \geq -7$$
$$\frac{-8x}{-8} \leq \frac{-7}{-8}$$
$$x \leq \frac{7}{8}$$

(Notice above how the inequality reversed direction when we divided both sides by a negative number.)

The domain of F is $\left\{x \middle| \; x \leq \frac{7}{8}\right\}$, "the set of all real x such that x is less than or equal to seven-eighths."

Extended Example 4a

Find the domain of the function $g(x) = \sqrt{2x-10}$.

RELATIONS AND FUNCTIONS

EXAMPLE G

Find the domain of the function $Q(t) = 2t^2 - 3t + 5$.

This time, there's no need to restrict the domain. Any real number can be squared, multiplied times negative three, and more. The domain consists of all real numbers.

Things to notice about this example:

- Notice that this time the independent variable (from the domain) is t rather than x. Any letter can do x's job.
- This function is a second-degree polynomial function, since it's defined by a polynomial. Second-degree polynomial functions are called **quadratic functions** (we'll look at those more in another chapter). Linear functions are first-degree polynomial functions.
- Every polynomial function has the entire set of real numbers for its domain.

When you define a function, you can specify precisely how you wish to restrict its domain.

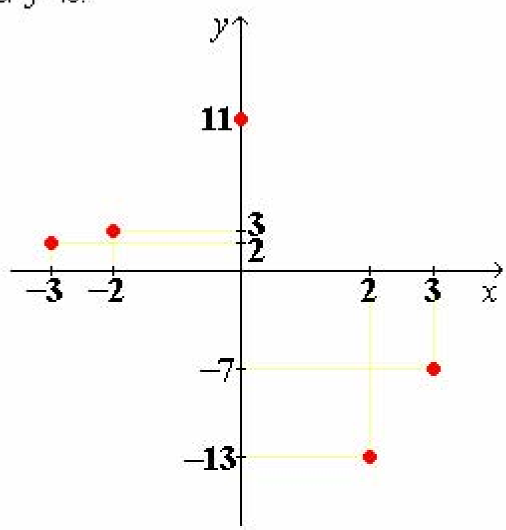
EXAMPLE H

Let $f(x) = \dfrac{x+11}{1-x}$, with domain $\{-3,-2,0,2,3\}$. Find the range of f, and graph this function.

We must evaluate f at each input in its specified domain. The resulting outputs constitute the range of f. Look at this table to study the calculations that lead to the range and points shown below.

The range of f is $\{-13,-7,2,3,11\}$.

The graph of f is:



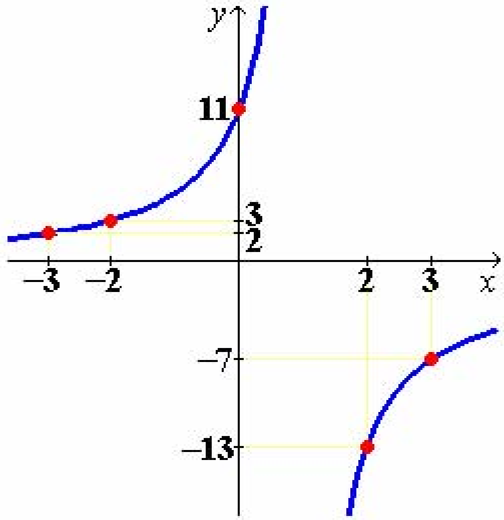
RELATIONS AND FUNCTIONS

When we don't explicitly restrict its domain as we did in the last example, the domain of

$$f(x) = \frac{x+11}{1-x}$$

includes all real numbers (except $x = 1$, which makes the denominator 0).

Plotting a few hundred more points for this function results in the curves shown in the graph below. (In this graph, you can still see the points from the previous screen.)



To graph any function by hand, you can simply plot enough points to enable you to "see" where the curve goes, and then smoothly connect the dots. But, as you might imagine, computers and graphing calculators are far more efficient at such tasks than humans!

Extended Example 5a

Let $p(x) = \dfrac{3x}{5x-2}$ with domain $\{-2,-1,0,1,2\}$. What is the range of p ?

EXAMPLE H

A store has 200 stereos for sale. The price at which the stereos will be sold is related to the number of stereos that are sold by the function $P(S) = 550 - 2S$, where P is the price of a stereo and S is the number of stereos sold. What is the domain of this function?

Although the function $P(S) = 550 - 2S$ is a linear function with all real numbers for its domain, we are told there are only 200 stereos. Also, there can't be a negative number of stereos, so the number sold must be between 0 and 200.

Further, the store cannot sell a fraction of a stereo, so we know the number sold must be a whole number.

Therefore, the domain of this function is the set of all whole numbers between 0 and 200, inclusive:

$$\left\{ S \,\middle|\, S \text{ is a whole number and } 0 \le S \le 200 \right\}.$$

EXAMPLE I

Suppose the number of ovens that can be assembled at a factory in one day is 8 less than four times the number of employees working on that day. Find a function that expresses the number of ovens that can be produced in a day as a function of the number of employees working. Use this function to calculate how many ovens can be produced with 12 employees.

Call the function N for "the number of ovens that can be assembled at the factory in one day," and the input variable w for "the number of workers." Let's translate from English into algebra:

"The number of ovens that can be assembled at the factory in one day is 8 less than four times the number of employees working on that day..." becomes:
$N = 4w - 8$.

We can think of N as a function of w. Using function notation, this equation can be written as:

$$\boxed{N(w) = 4w - 8}$$

Inputting $w = 12$ to this function gives us:

$$N(w) = 4 \cdot w - 8$$
$$N(12) = 4 \cdot 12 - 8$$
$$= 48 - 8$$
$$N(12) = 40$$

40 ovens can be assembled by 12 employees.

RELATIONS AND FUNCTIONS

Extended Example 6a

A salesperson earns $80 per day, plus a 9% sales commission. Find a function that expresses her earnings as a function of sales, and use it to compute her earnings if the total sales were $600.

END OF LESSON

For $f(x) = \dfrac{x-2}{x+1}$, find x so that $f(x) = 2$.

Evaluate the function at the indicated value.

$Q(t) = \dfrac{3-2t}{t^2}$ **at** $t = -3$

For $g(x) = x^2 - 7x + 2$, find the range if the domain is $\{-4, -2, 0, 1, 4\}$.

If the cost in dollars to mail a package which weighs w ounces is given by the function $C(w) = 2.75 + 0.65w$, how much would it cost to mail a package which weighs 14 ounces?

GRAPHING LINEAR INEQUALITIES IN TWO VARIABLES

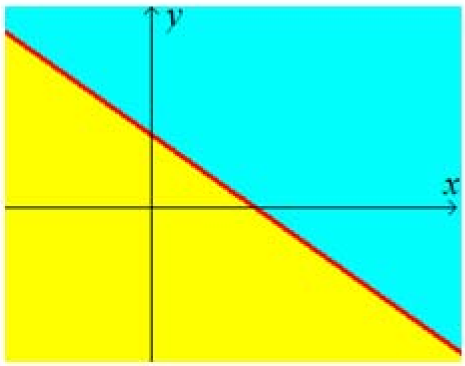
Introduction

In this lesson, you will learn how to graph **linear inequalities** in two variables. These inequalities have solutions that consist of entire regions of the x,y plane. Linear inequalities are applied to many different fields, including operations research, where finding an optimal point in a feasible region amounts to maximizing a profit or minimizing a cost.

GRAPHING LINEAR INEQUALITIES IN TWO VARIABLES

Dividing the Plane

If you draw a line on a plane, it divides the plane into three distinct parts: all the points on one side of the line, all the points on the other side of the line, and the points on the line itself.



The solution set of a linear inequality in two variables consists of all the points on one side of a line. It may or may not include the line itself.

We solve a linear inequality by graphing it in two steps. The first step is to graph the correct line; the second step is to shade the appropriate side of that line. All the points on the shaded side of the line satisfy the inequality. When you substitute the x, y coordinates of the points in the shaded region into the inequality, the inequality is true. The inequality is not true for the points on the opposite, un-shaded side of the line.

Now let's look at an example so you can see what this is all about!

GRAPHING LINEAR INEQUALITIES IN TWO VARIABLES

EXAMPLE A

Graph $y \le 2x + 1$.

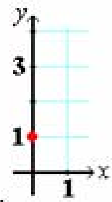
Step 1: Replace the inequality with an equal sign and graph the resulting equation. Graph $y = 2x + 1$. This equation is in slope-intercept form:

$$y = mx + b$$

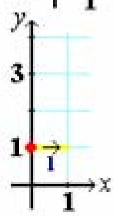
$$y = 2x + 1$$

The slope is 2 and the y-intercept is the point $(0, 1)$.

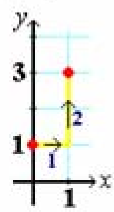
Plot the y-intercept, $(0, 1)$:



Since the slope = 2 = $\frac{2}{1} = \frac{\text{rise}}{\text{run}}$, you run a distance of 1 (to the right)...



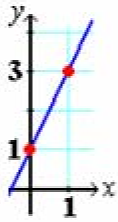
...and then you rise up a distance of 2 to get back to the line:



continued...

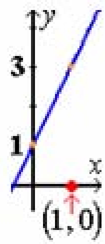
GRAPHING LINEAR INEQUALITIES IN TWO VARIABLES

Example A, continued…



In this case, the inequality is ≤ (less than or equal to). "Or equal to" means that the line itself is included in the solution. We indicate this by drawing a **solid line** through the points.

Step 2: We must decide which side of the line is included in the solution set (we will shade that side of the line). Pick a point that is not on the line with coordinates that are easy to operate with (small numbers, preferably 0 or 1). For example, we'll pick the point $(1,0)$:



This is our **test point**. Do the coordinates of this point make the inequality true? We can answer this important question by substituting the coordinates of the point into the original inequality:

$$(x,y)=(1,0) \quad \rightarrow \quad y \le 2\cdot x+1$$

$$0 \le 2\cdot 1+1$$

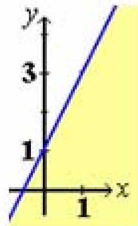
$$0 \le 3 \quad \textit{True!}$$

continued…

GRAPHING LINEAR INEQUALITIES IN TWO VARIABLES

Example A, continued ...

Since our test point $(1,0)$ satisfies the inequality, we shade the area on the side of the line that contains it:



$y \le 2x + 1$

The x,y coordinates of every point in the shaded region satisfy the linear inequality $y < 2x + 1$. In other words, the y-coordinate of each point in that region is less than one more than twice its x-coordinate. Along the line itself, each point's y-coordinate is exactly one more than twice its x-coordinate, which also satisfies the inequality since it is "or equal to" $2x + 1$. However, on the un-shaded side of the line, y is greater than $2x + 1$, and none of those points will satisfy the given inequality.

Words to the Wise

- If a test point does <u>not</u> satisfy the inequality, then shade the opposite side of the line.
- NEVER pick a test point that's on the line.
- The usual (and easiest) test point to use is the origin $(0,0)$.

GRAPHING LINEAR INEQUALITIES IN TWO VARIABLES

Extended Example 1a

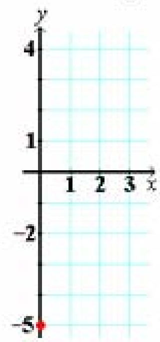
Graph $y \geq -x + 1$.

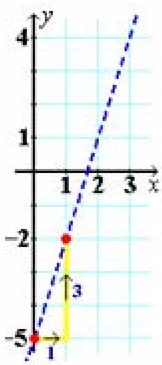
EXAMPLE B

Graph $y > 3x - 5$.

Step 1: Replace the inequality symbol with an equal sign and graph the resulting equation. Graph $y = 3x - 5$. This equation is in slope-intercept form. We know the slope is 3 and the y-intercept is the point $(0, -5)$.

Plot the y-intercept $(0, -5)$:





Since the slope = $3 = \frac{3}{1} = \frac{\text{rise}}{\text{run}}$, when you run a distance of 1 (to the right), you must rise up a distance of 3 to get back to the line.

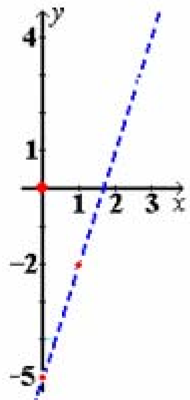
Note: This time, we graph this as a **dashed line** because the solution does not include the points on the line, since this is a strict inequality (>).

continued...

GRAPHING LINEAR INEQUALITIES IN TWO VARIABLES

Example B, continued...

Step 2: We must decide which side of the line includes the solution set (we will shade that side of the line). Again, pick a point that is clearly on one side of the line and that has coordinates that are easy to operate with (small numbers, preferably 0 or 1). For example, we'll pick the origin $(0,0)$:



Do the coordinates of this point make the inequality true? We can answer this important question by substituting the coordinates of the point into the original inequality:

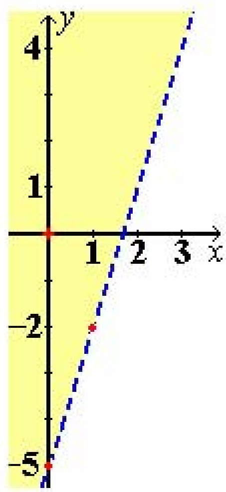
$$(x,y)=(0,0) \quad \rightarrow \quad y>3\cdot x-5$$
$$0>3\cdot 0-5$$
$$0>-5 \quad \textit{True!}$$

continued...

GRAPHING LINEAR INEQUALITIES IN TWO VARIABLES

Example B, continued ...

Since our test point $(0,0)$ satisfies the inequality, we shade the side with $(0,0)$ because these points are included in the solution set.



$y > 3x - 5$

GRAPHING LINEAR INEQUALITIES IN TWO VARIABLES

Extended Example 2a

Graph $y > -\frac{1}{5}x + 2$.

GRAPHING LINEAR INEQUALITIES IN TWO VARIABLES

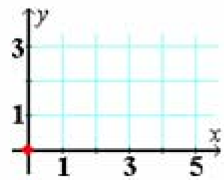
Sometimes we can't use the origin for a test point. This occurs whenever the line passes through the origin.

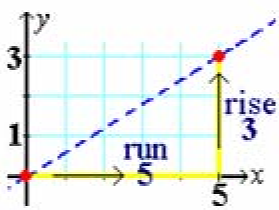
EXAMPLE C

Graph $y > \frac{3}{5}x$.

Step 1: Replace the inequality with an equal sign and graph the resulting equation. Graph $y = \frac{3}{5}x$. This equation is in slope-intercept form, so we know the slope is $\frac{3}{5}$ and the y-intercept is the point $(0, 0)$. The line passes through the origin.

Plot the y-intercept, $(0, 0)$:





Since the slope $= \frac{3}{5} = \frac{\text{rise}}{\text{run}}$, we run a distance of 5 (to the right), then rise up a distance of 3 to get back to the line.

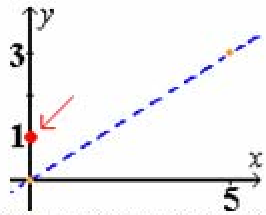
We use a dashed line, since we do not include the points on the line. (The inequality symbol is " > ".)

continued...

GRAPHING LINEAR INEQUALITIES IN TWO VARIABLES

Example C, continued...

Step 2: We must decide which side of the line to shade. Pick a point with easy coordinates (small numbers, preferably 0 or 1), for which you can clearly see on which side of the line the point sits. Since we can't use the origin, we'll use point $(0,1)$:



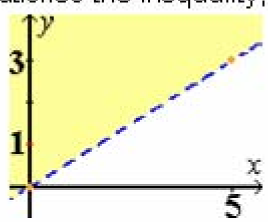
Do the coordinates of this point make the inequality true? We can answer this important question by substituting the coordinates of the point into the original inequality:

$$(x,y) = (0,1) \quad \rightarrow \quad y > \frac{3}{5} \cdot x$$

$$1 > \frac{3}{5} \cdot 0$$

$$1 > 0 \quad \textit{True!}$$

Since our test point satisfies the inequality, we shade its side of the line.



$$\boxed{y > \frac{3}{5}x}$$

EXAMPLE D

Graph $4x+3y \le 12$.

Step 1: Replace the inequality with an equal sign and graph the resulting equation. Graph $4x+3y=12$. This time the equation of the line is <u>not</u> in slope-intercept form. We can either put it into slope-intercept form and proceed as we did in previous examples, or, we can find two points on the graph. For this problem, it's easiest to find the x and y-intercepts without putting the equation into slope-intercept form first.

To find the x-intercept, we set $y=0$:

$$4x+3\cdot y=12$$
$$4x+3\cdot 0=12$$
$$4x=12$$
$$x=3$$

The x-intercept is the point $(3,0)$.

To find the y-intercept, we set $x=0$:

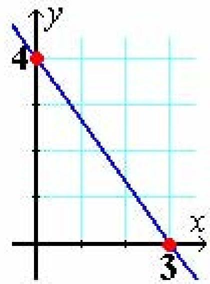
$$4\cdot x+3y=12$$
$$4\cdot 0+3y=12$$
$$3y=12$$
$$y=4$$

The y-intercept is the point $(0,4)$.

continued...

Example D, continued…

After plotting these two intercepts, we can graph the line. The line will be solid, due to the "≤" symbol:



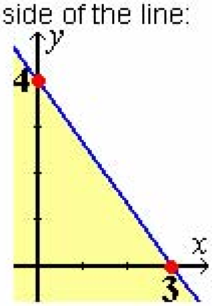
Step 2: We must decide which side of this line to shade. Again, we pick our old friend the origin, $(0,0)$. Is the origin in the solution set? We answer this by substituting zeros into the original inequality:

$$(x, y) = (0, 0) \quad \rightarrow \quad 4 \cdot x + 3 \cdot y \le 12$$

$$4 \cdot 0 + 3 \cdot 0 \le 12$$

$$0 \le 12 \qquad \textit{True!}$$

So we shade the origin side of the line:



$$\boxed{4x + 3y \le 12}$$

Extended Example 3a

Graph $5x - 2y \geq 10$.

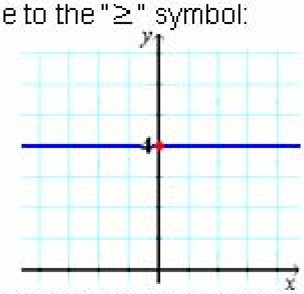
GRAPHING LINEAR INEQUALITIES IN TWO VARIABLES

EXAMPLE E

Graph $3y \geq 12$.

Step 1: Replace the inequality with an equal sign and graph the resulting equation. We graph the line $3y = 12$. Dividing both sides by 3, we get $y = 4$.

Recall that this is the form of the equation of a horizontal line. In this case, the y-intercept is at the point $(0,4)$. Graph this horizontal line. The line will be solid due to the "$\geq$" symbol:



Step 2: We must decide which side of this line to shade. Again, we can choose the origin $(0,0)$. Is the origin in the solution set? We answer this by substituting a zero into the original inequality:

$$(x,y) = (0,0) \quad \rightarrow \quad 3 \cdot y \geq 12$$

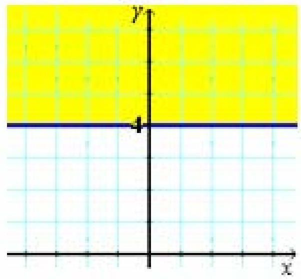
$$3 \cdot 0 \geq 12$$

$$0 \geq 12 \quad \textit{False!}$$

continued...

Example E, continued...

Since the origin made a false statement, the side that does not include the origin contains our solution set:

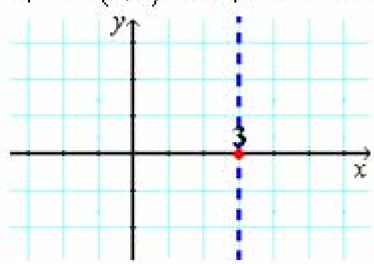


$3y \geq 12$

EXAMPLE F

Graph $2x < 6$.

Step 1: Replace the inequality with an equal sign and graph the resulting equation: We must graph the line $2x = 6$. Dividing both sides by 2, we get $x = 3$. Recall that this is the form of the equation of a vertical line. In this case, the x-intercept is at the point $(3, 0)$. Graph this vertical line. It is dashed due to the "<" symbol.



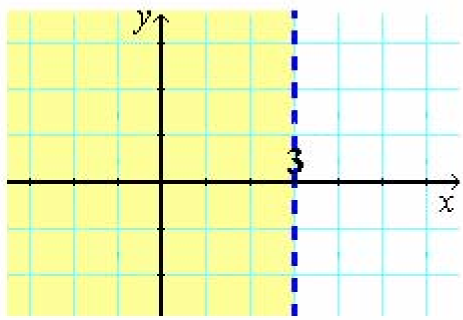
continued...

Example F, continued...

Step 2: We must decide which side of this line to shade. Again, we can pick our old friend the origin $(0,0)$. Is the origin in the solution set? We answer this by substituting a zero into the original inequality:

$$(x,y)=(0,0) \quad \rightarrow \quad 2\cdot x<6$$
$$2\cdot 0<6$$
$$0<6 \quad \textit{True!}$$

Therefore, we shade the side with the origin.



$\boxed{2x<6}$

Graph: $y > -2x + 3$

Graph: $4 - y - x \geq 0$

Graph: $y < -2$

Graph the linear inequality.

$y < \frac{4}{3}x - 2$

Graph the linear inequality.

$3x - 4y < 24$

Graph the linear inequality.

$-2x - 7y \le 28$